UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2020.

CÁLCULO DIFERENCIAL
EXAMEN # 1 (FORMA REMOTA). — BB
FECHA: **SÁBADO 6 DE JUNIO DE 2020: 14:30 HORAS**

Nombre: ANSWER KEY

Instrucciones:

- El examen consta de **SEIS** problemas con diferentes puntajes.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL.** Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y calculadora sencilla. Cite cuando use libros o apuntes, o su calculadora.
- Para recibir puntaje: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE** y **JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla en su escaneado.

(1) **(10 puntos.)** Calcule la ecuación de la recta tangente a la gráfica de la función $g(x) = -\dfrac{1}{(17-x)^{1/2}}$ en el punto $(1, -1/4)$. Use la definición de derivada.

(2) Calcule:
   (a) **(10 puntos)** la derivada de $g(x) = \cos(\tan^2(\sin 5x))$.
   (b) **(10 puntos)** $(\cos x)^{(195)}$.

(3) **(10 puntos.)** Encuentre la ecuación de la recta ortogonal a la curva $x^4 + y^4 = 32$ en el punto $(2, -2)$.

(4) **(20 puntos.)** Un automóvil se mueve a lo largo de una carretera y su posición está dada por la función $y(t) = t^4 - 8t^3$. ¿En qué instantes el carro acelera?

(5) **(20 puntos.)** Un pájaro vuela horizontalmente a 20 metros sobre usted a una velocidad de 8 metros por segundo. ¿A qué velocidad angular (respecto a la horizontal) usted va moviendo sus ojos al mirar el pájaro alejarse cuando el pájaro está a 30 metros de usted?

(6) **(20 puntos.)** La gráfica (mostrada en el otro archivo que est dentro de este examen) muestra la cantidad, $C(t)$, acumulada de infectados a lo largo de varios días por el virus SARS-CoV2 en algún país. Como se hizo en clase, grafique la derivada, $C'(t)$, y diga qué significa esta gráfica. Ahora explique qué quiere decir *"aplanar la curva"* y dibuje otras dos gráficas, una para $C(t)$ y otra para $C'(t)$, en donde *"las aplanó"*. (Haga, en la medida de lo posible, las gráficas con la misma escala).

(*) **FÓRMULAS.**
   (a) $\sin^2\alpha + \cos^2\alpha = 1$.
   (b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$
   (c) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
   (d) $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$.

UAM - Azcapotzalco $\boxed{\text{Cálculo Diferencial}}$ Saturday, June 6, 2020

Examen # 1 Answer Key

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(1) We have the function $f(x) = -\frac{1}{\sqrt{17-x}}$ and $f(1) = -\frac{1}{4}$, so that the graph passes through $(1, -1/4)$. It remains to find the slope of the tangent line, using the derivative function: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Using this definition:

$$f'(x) = \lim_{h \to 0} \frac{\left(\frac{-1}{\sqrt{17-(x+h)}}\right) - \left(\frac{-1}{\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{-\sqrt{17-x} + \sqrt{17-(x+h)}}{\sqrt{17-(x+h)}\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\sqrt{17-(x+h)} - \sqrt{17-x}}{\sqrt{17-(x+h)}\sqrt{17-x}} \frac{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{(17-(x+h)) - (17-x)}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-h}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\left(\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)\right)}$$

Now, we can directly substitute $h = 0$; to evaluate limit:

$$= \frac{-1}{\sqrt{17-x}\sqrt{17-x}\left(\sqrt{17-x} + \sqrt{17-x}\right)}$$

$$= \frac{-1}{(17-x)\, 2\sqrt{17-x}}$$

$= 1 =$

I.e., the derivative function is:

$$f'(x) = \frac{-1}{2(17-x)^{3/2}}$$

At the value $x = 1$, we get the slope of the tangent line:

$$m = f'(1) = \frac{-1}{2(16)^{3/2}} = \frac{-1}{2(4^3)} = \frac{-1}{2^7}.$$

i.e. $m = \frac{-1}{128}$. And the equation of the tangent line in point-slope form is

$$y - \left(\frac{-1}{4}\right) = \frac{-1}{128}(x-1)$$

i.e. $$\boxed{y + \frac{1}{4} = \frac{-1}{128}(x-1)}$$

In slope-y-intercept form:

$$y = \frac{-1}{128}x + \frac{1}{128} - \frac{1}{4} = \frac{-1}{128}x + \frac{1-32}{128}$$

$$\boxed{y = \frac{-1}{128}x - \frac{31}{128}}$$

(2) (a) Compute the derivative of.

$$g(x) = \cos(\tan^2(\sin 5x)).$$

We use the Chain rule:

$$\frac{dg}{dx} = -\sin(\tan^2(\sin 5x))\frac{d}{dx}\left(\tan^2(\sin(5x))\right)$$

$$= -\sin(\tan^2(\sin(5x)))\, 2\tan(\sin 5x)\frac{d}{dx}\left(\tan(\sin(5x))\right)$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin(5x))\frac{d}{dx}\sin(5x)$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)\frac{d}{dx}(5x)$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)\cdot 5$$

$$\boxed{\frac{dg}{dx} = -10\sin(\tan^2(\sin(5x)))\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)}$$

(b) Compute $(\cos x)^{195} = \frac{d^{195}}{dx^{195}}(\cos x)$

Since 192 is a multiple of 4:

$$\frac{d^{192}}{dx^{192}}(\cos x) = \cos x.$$

Then $\frac{d^{195}}{dx^{195}}\cos x = \frac{d^3}{dx^3}\left(\frac{d^{192}}{dx^{192}}\cos x\right) = \frac{d^3}{dx^3}(\cos x)$

$$= \frac{d^2}{dx^2}(-\sin x) = \frac{d}{dx}(-\cos x) = (-1)(-1)\sin x:$$

$$\boxed{\frac{d^{195}}{dx^{195}}(\cos x) = \sin x}$$

(3) The orthogonal line to $x^4+y^4=32$ at $(2,-2)$ is computed as follows.

First, check that $(2,-2)$ is on the curve..

$x^4+y^4=(2)^4+(-2)^4 = 16+16=32$ ✓ It does.

Now, compute implicitly the derivative, by using the Chain rule:

$$\frac{d}{dx}(x^4+y^4)=\frac{d}{dx}(32) \Rightarrow 4x^3+4y^3\frac{dy}{dx}=0.$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x^3}{y^3}}$$ is the implicit derivative.

At $(2,-2)$, the ~~derivative~~ tangent line has slope

$$m_{tan} = \frac{dy}{dx}\Big|_{(2,-2)} = -\frac{(2)^3}{(-2)^3} = \frac{2^3}{2^3} = 1.$$

Then, the orthogonal line has the slope.

$$m = \frac{-1}{m_{tan}} = \frac{-1}{1} = -1$$

Hence, the equation in point-slope form is:

$$y-(-2) = m(x-(2)) \qquad \boxed{y+2=-(x-2)}$$

or in the form slope-y-intercept:

$$y = x-2-2 \qquad \boxed{y=-x}$$

(4) The function position of the automobile is

$$y(t) = t^4 - 8t^3$$

We compute the velocity: $v = y'(t) = 4t^3 - 24t^2$

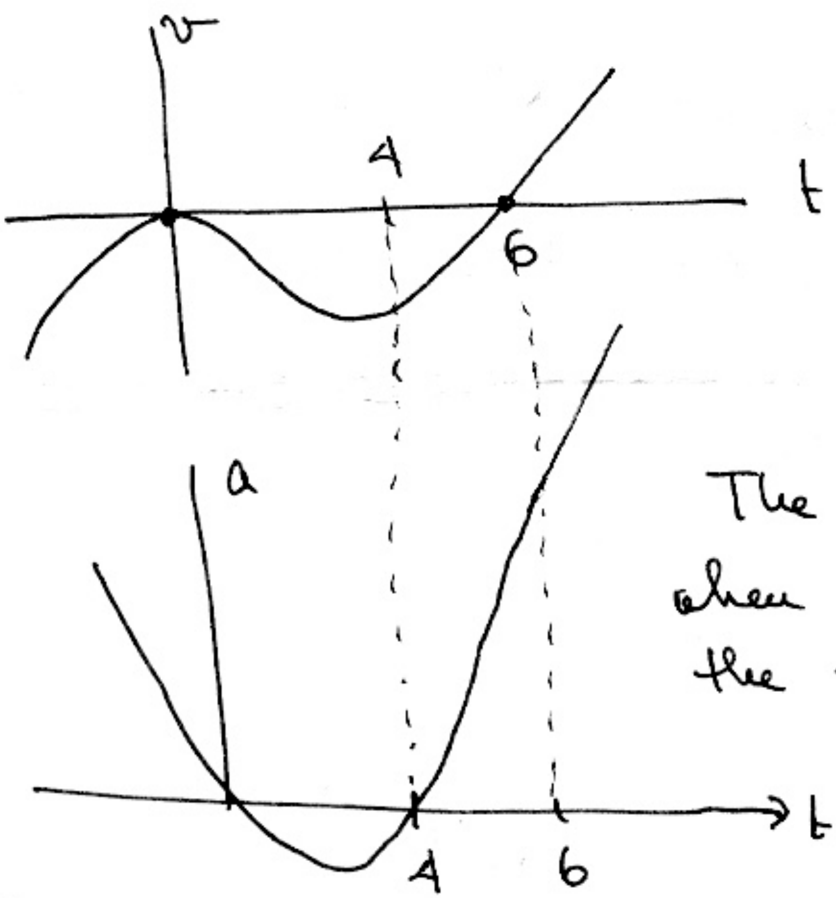
and acceleration: $a = y''(t) = 12t^2 - 48t$.

We can factor the velocity and acceleration.

$$v = 4t^2(t-6)$$

$$a = 12t(t-4)$$

We can plot v and a vs. t:



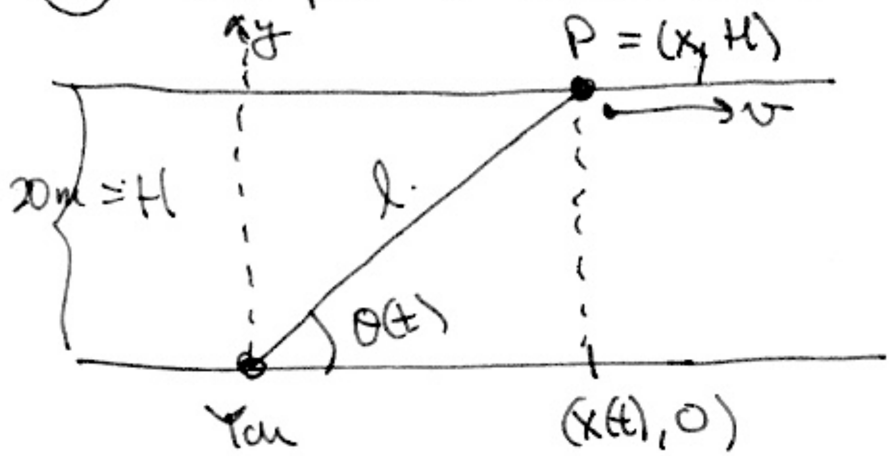
The car speeds up when a and v have the same sign.

Notice that $v > 0$ and $a > 0$ on $(6, \infty)$

and $v < 0$ and $a < 0$ on $(0, 4)$

Then the car speeds up for: $\boxed{t \in (0,4) \cup (6,\infty)}$

(5) We plot a scheme of the situation:



P is the position of the bird, $(x, H) = (x(t), H)$

H is a constant.

$x(t)$ = function of t.

$\theta = \theta(t)$ = angle with respect to the horizontal.

$v = \dot{x}(t)$ is the velocity of bird.

$\theta = \theta(t)$ is also a function of t.

$l = l(t)$ is the distant from you to the bird.

We want $\frac{d\theta}{dt}$, the angular velocity of your eyes.

Then, $\tan(\theta(t)) = \frac{H}{x(t)}$ Computing the derivative on both side.

$$\sec^2(\theta(t)) \frac{d\theta}{dt} = -\frac{H}{x^2}\frac{dx}{dt}$$

Since $\sec^2\theta = \frac{1}{\cos^2\theta} = \left(\frac{l}{x}\right)^2$ then we have:

$$\left(\frac{l}{x}\right)^2 \frac{d\theta}{dt} = -\frac{H}{\cancel{x}^2}\frac{dx}{dt} \Rightarrow \boxed{\frac{d\theta}{dt} = -\frac{H}{l^2}\frac{dx}{dt}}$$

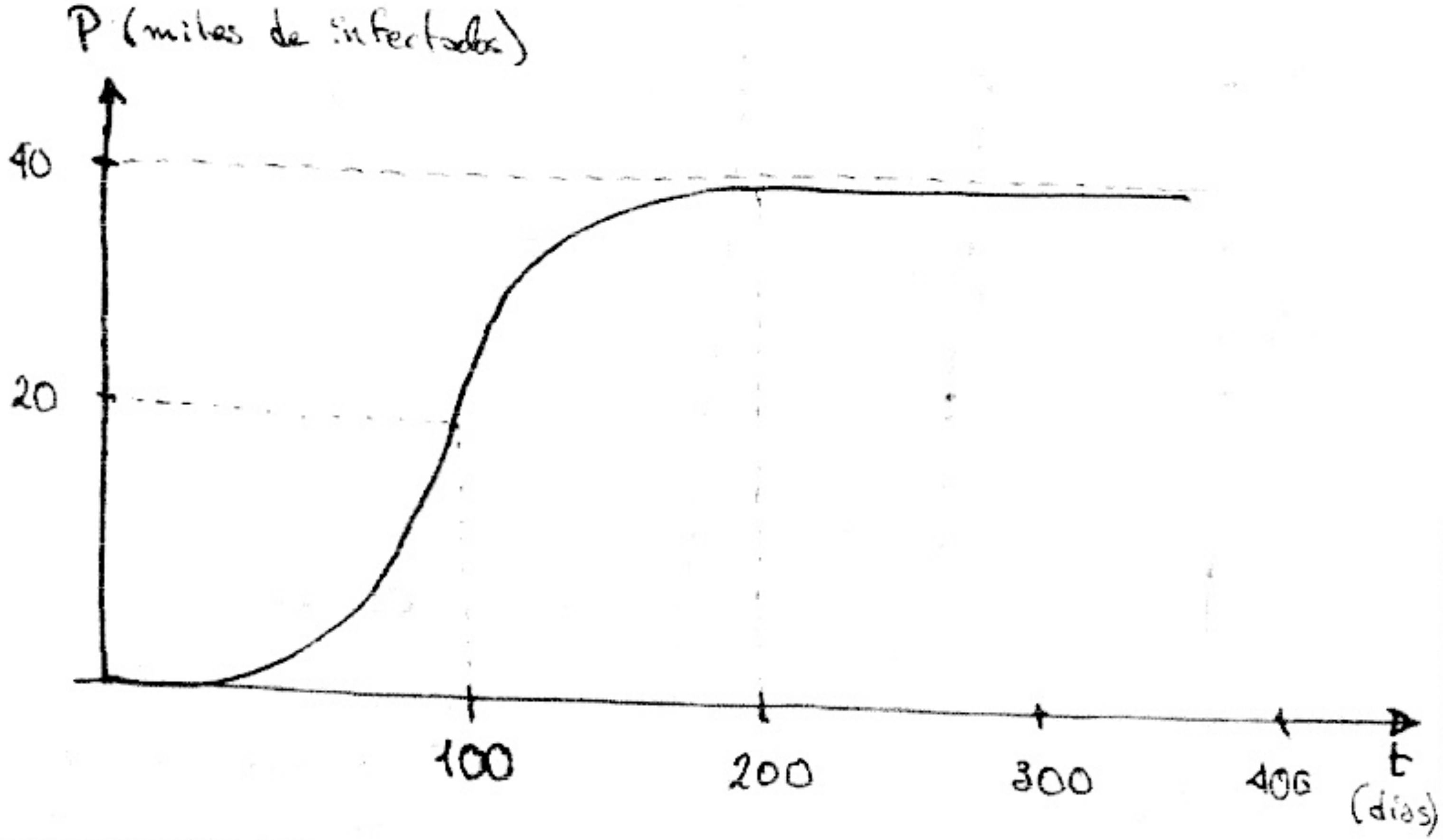
Related rates

When $l = 30$ m, the bird is 30 m. away, and since $H = 20$ m, $\frac{dx}{dt} = v = 6$ m/sec, we have

$$\frac{d\theta}{dt} = -\frac{20\,m}{30^2 m^2} \cdot \frac{6\,m}{sec} = -\frac{120}{900}\frac{1}{sec} = -\frac{4}{30}\frac{1}{sec}$$

$$\frac{d\theta}{dt} \approx -0.1333\ 1/sec \qquad \boxed{\frac{d\theta}{dt} = -\frac{2}{15}\frac{1}{sec}}$$

Problem ⑥. Examen #1 — BB.



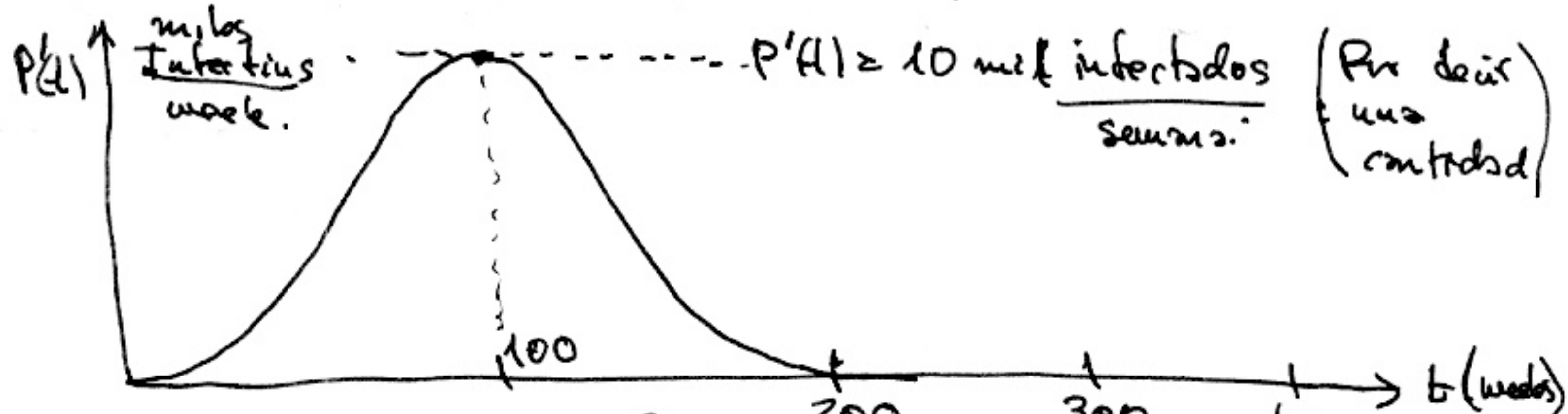
Solution Problem ⑥.

(a) We observe that at $t=0$ and after $t=200$, $P'(t)=0$.

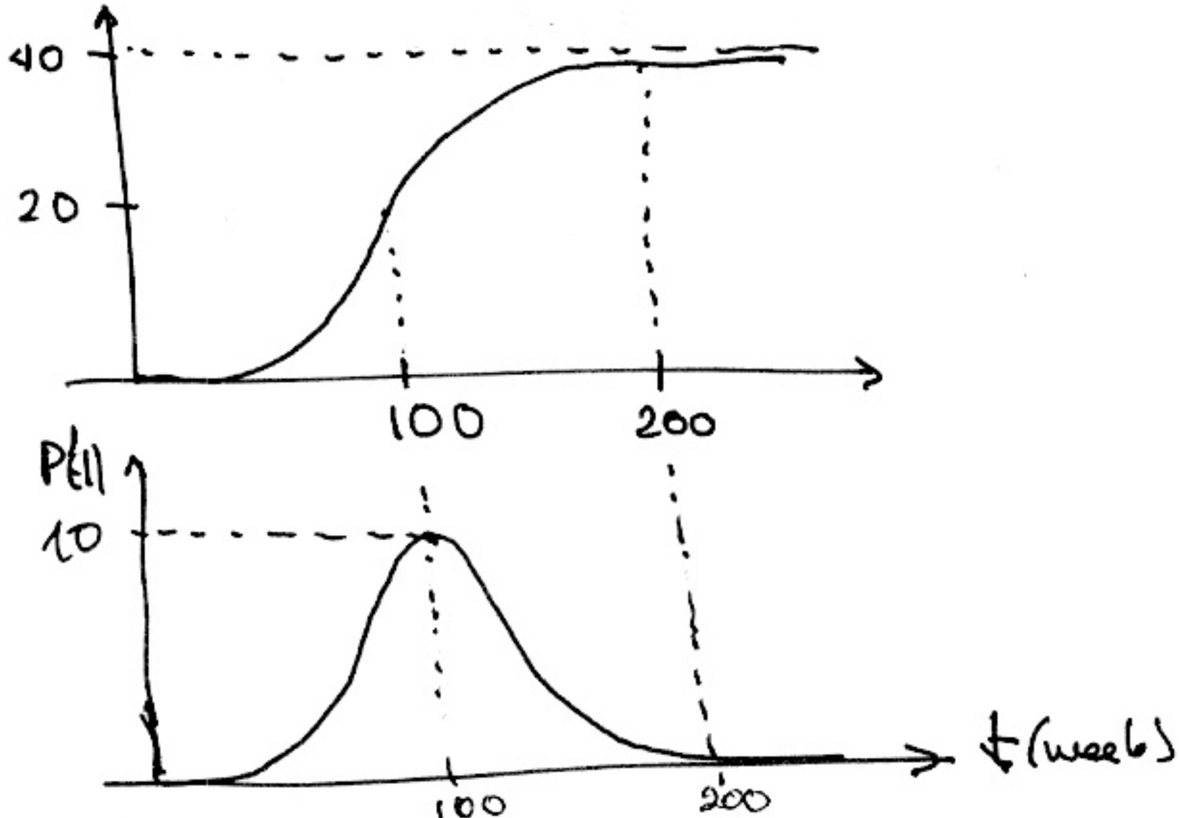
(b) $P'(t)$ is small at $t\approx 0$ and increases up to $t=100$.

(c) $P'(t)$ is máx at $t=100$.

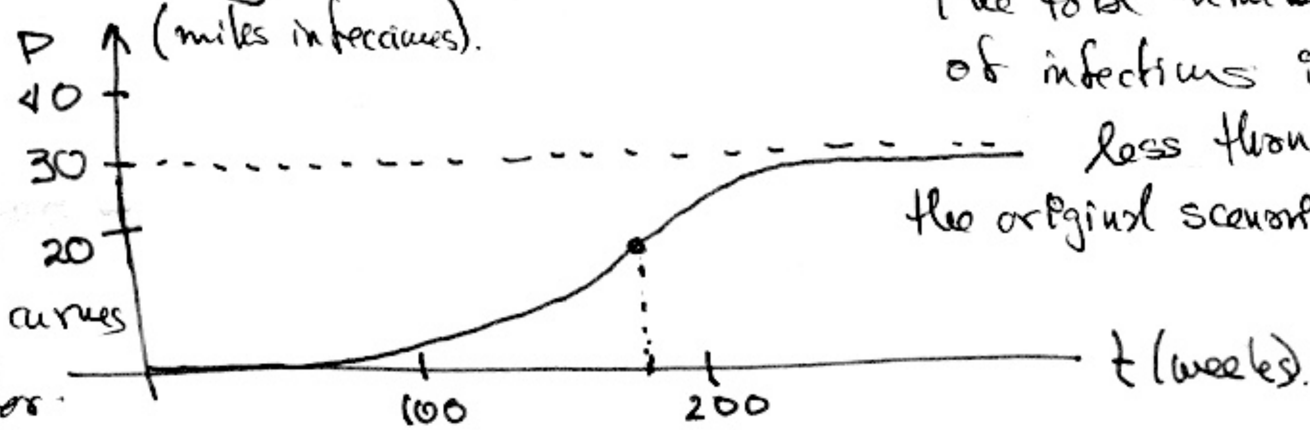
(d). $P'(t)$ decreases on $(100, 200)$, until $P'(t)=0$.



Plotting the graphs again.

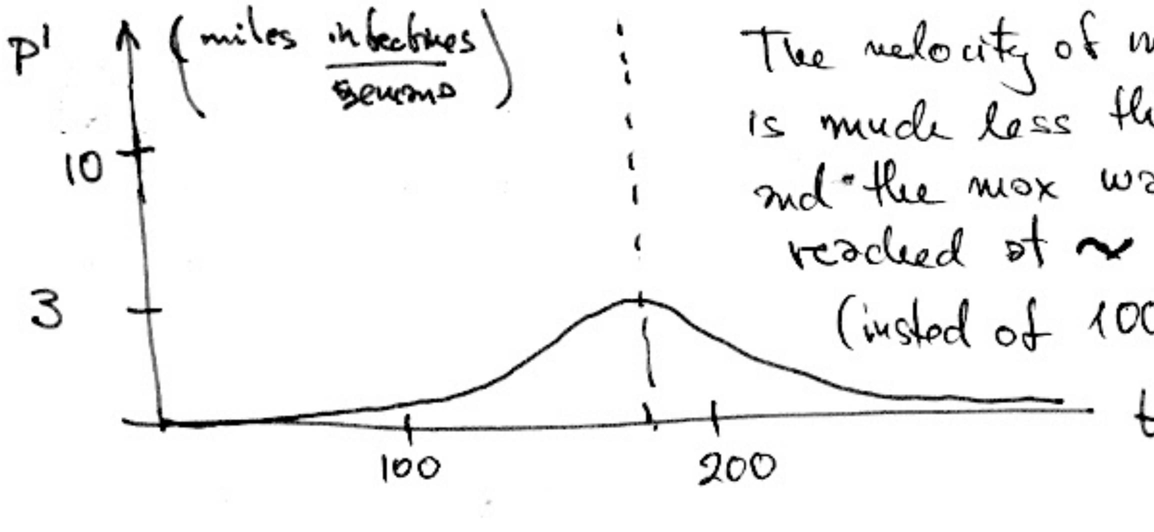


By "flattening" the curve we mean:



The total number of infections is less than in the original scenario.

These curves are flatter.



The velocity of infection is much less than 10, and the max was reached at ~ 190 weeks (instead of 100 weeks).