UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2020.
CÁLCULO DIFERENCIAL
EXAMEN # 2 (FORMA REMOTA). FECHA: SÁBADO 27 DE JUNIO DE 2020: HORA 14:30. HORA DE ENTREGA: 16:30

Nombre: KEY

- El examen consta de **CUATRO** problemas con diferentes puntajes. Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- *Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo.* Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- Para recibir puntaje: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla con su examen escaneado.

(1) (**35 puntos**) Bosqueje (es decir, dibuje a mano) la gráfica de la siguiente función. Para ello, detalle, calcule y argumente lo que enseguida se pide. Con base a esta información grafique la función. Compare con la gráfica que le de su graficador. Incluya imagen de la gráfica dada por el graficador.

$$f(x) = 2\sqrt[3]{x^2(3-x)}$$

(a) Dominio de f.
(b) Encuentre las intersecciones con los ejes.
(c) Encuentre las simetrías que pueda tener la gráfica y úselas posteriormente para el bosquejo.
(d) Comportamiento al infinito y asíntotas de la gráfica de f.
(e) Puntos críticos de f.
(f) Intervalos de monotonía de f.
(g) Máximos y mínimos locales y, en su caso, máximos y mínimos globales de f.
(h) Intervalos de convexidad.
(i) Puntos de inflexión (en caso de que existan).
(j) Verifique sus máximos y mínimos locales con el criterio de la segunda derivada.
(k) Bosqueje la gráfica.
(l) Compare con la gráfica de su graficador (incluya esta gráfica).

(2) **(35 puntos.)** Un contenedor rectangular **con tapa** debe contener un volumen de 10 m^3. La longitud de su base es dos veces su ancho. El material de las tapas y la base cuesta lo mismo por metro cuadrado, digamos 50 pesos por metro cuadrado. Encuentre las dimensiones de la caja para tener un costo mínimo. ¿Cuál es ese costo?

(3) **(10 puntos.)** Este problema es sobre el Teorema de Rolle.
  (a) Enuncie el Teorema de Rolle.
  (b) Escoja números fáciles a y b para definir el intervalo $[a, b]$. Escoja una función fácil $f(x)$ que cumpla las condiciones del Teorema de Rolle.
  (c) Verifique el Teorema de Rolle con el intervalo y función del inciso anterior y encuentre el valor de c en este caso.

(4) **(20 puntos.)**
  (a) ¿Qué ingeniería estudia?
  (b) Ahora enuncie un problema de maximización o minización que usted considere aparece en su ingeniería.
  (c) ¿Qué variable tiene que maximizar o minimizar? ¿Cuáles son sus variables independientes?
  (d) ¿Cómo resolvería el problema usando Cálculo Diferencial?

(1) We have to sketch the graph of the function:

$$f(x) = 2\sqrt[3]{x^2(3-x)} \quad \ldots\ldots \ (*)$$

Sometimes it will be useful to write

$$f(x) = 2\cdot(3x^2 - x^3)^{1/3} \quad \ldots\ldots \ (**)$$

(a) Domain. We have a $\sqrt[3]{\ }$, which is free for definition of the functions; there is no denominator, so there is no singularities, then:

$$\mathrm{Dom}(f) = \mathbb{R}.$$

(b). Intersections with axis

y-intercept: $y = f(0) = 2\sqrt[3]{0(3-0)} = 0.$ Eq'n (*).

Passes through $(0,0)$.

x-intercepts. Solve: $f(x) = 0$.

Use eq'n(*): $2\sqrt[3]{x^2(3-x)} = 0 \Rightarrow x^2(3-x) = 0$

$\Rightarrow x_1 = 0$ and $x_2 = 3 \Rightarrow \vec{x}_1(0,0)$ (already found)

$\vec{x}_2 = (3,0)$.

(c) Since $f(-x) \overset{(*)}{=} 2\sqrt[3]{(-x)^2(3-(-x))} = 2\sqrt[3]{x^2(3+x)}$

$\neq f(x)$

There is no symmetries w.r.t. axis.

There is no trig functions: No periodicity either.

(d). Using (**)

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2\left(3x^2-x^3\right)^{\frac{1}{3}}$$

$$= 2\lim_{x\to-\infty} x\left(\frac{3}{x}-1\right)^{\frac{1}{3}}$$

$$= 2\lim_{x\to-\infty} x \lim_{x\to-\infty}\left(\frac{3}{x}-1\right)^{\frac{1}{3}}$$

$$= 2\left(\lim_{x\to-\infty} x\right)(-1)$$

$$= 2(-\infty)(-1) = +\infty$$

$$\lim_{x\to+\infty} f(x) = 2\left(\lim_{x\to+\infty} x\right)\lim_{x\to\infty}\left(\frac{3}{x}-1\right)^{\frac{1}{3}}$$

$$= 2\left(\lim_{x\to+\infty} x\right)(-1)$$

$$= -\infty$$

There is no horizontal asymptotes.

However: $\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{1}{x}\left[2x\left(\frac{3}{x}-1\right)^{\frac{1}{3}}\right]$

$$= 2\lim_{x\to+\infty}\left(\frac{3}{x}-1\right)^{\frac{1}{3}}$$

$$= -2$$

Same limit for $x\to-\infty$..

Then, there is a slant asymptote with slope $m=-2$.

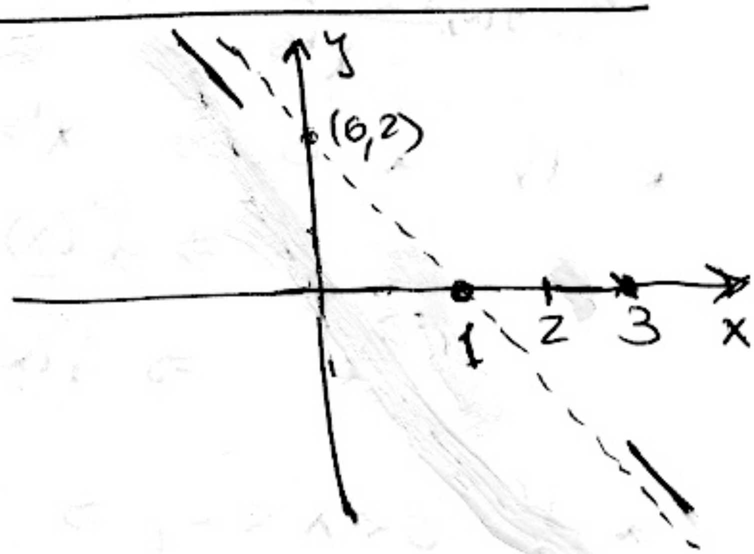
$\boxed{060627\text{-}2020.}$

to find the y-intersept:

or $\lim\limits_{x\to\infty}\left(f(x)-mx\right) = \lim\limits_{x\to\infty}\left(2x\left(\frac{3}{x}-1\right)^{\frac{1}{3}}+2x\right)$

$$= \lim_{x\to\infty} 2x \cdot \frac{\left(\frac{3}{x}-1\right)-1}{\left[\left(\frac{3}{x}-1\right)^{2/3}-\left(\frac{3}{x}-1\right)^{1/3}+1\right]} = \lim_{x\to\infty}\frac{2x\left(\frac{3}{x}\right)}{\left(\left(\frac{3}{x}-1\right)^{2/3}-\left(\frac{3}{x}-1\right)^{1/3}+1\right)} = \frac{6}{3}$$

Then, the oblique asymptote. $\boxed{y=-2x+2}$



We have a partial graph of f(x).

(e) To find the critical points, we require the first derivative. Use eq'n (**) and the Chain rule:

$$\frac{df}{dx} = 2\frac{d}{dx}\left(3x^2-x^3\right)^{1/3} = \frac{2}{3}\left(3x^2-x^3\right)^{-2/3}\left(6x-3x^2\right)$$

$$= \frac{2}{3}\,\frac{3(2-x)x}{\left(3x^2-x^3\right)^{2/3}} = \frac{2\,(2-x)x}{\left(x^2(3-x)\right)^{2/3}} = \frac{2(2-x)x}{x^{4/3}(3-x)^{2/3}}$$

i.e. $\boxed{\frac{df}{dx} = \frac{2(2-x)}{x^{1/3}(3-x)^{2/3}}}$

(i) If $f'(x)=0$, then $2(2-x)=0 \Rightarrow \boxed{x_1=2}$ } Horizontal tangent at

(ii) $f'(x)$ does not exist when $x^{1/3}(3-x)^{2/3} \Rightarrow$ $\boxed{x_2=0 \quad x_3=3}$
f has a cusp or vertical tangent.

(f) Intervals of monotony are $(-\infty,0)$, $(0,2)$, $(2,3)$, ...

We have: $\frac{df}{dx} = \frac{2(2-x)}{x^{1/3}(3-x)^{2/3}}$.

Observe that $2>0$, and $(3-x)^{2/3}>0$ on all these intervals. Then, the sign of the derivative is determined by $\tilde{f}(x) = \frac{2-x}{x^{1/3}}$

* If $x<0 \Rightarrow x^{1/3}<0$ and $2-x>0$

$$\Rightarrow \tilde{f}(x)<0 \Rightarrow f'(x)<0$$

$$\Rightarrow f(x) \searrow \text{ on } (-\infty,0).$$

* If $0<x<2$, $\Rightarrow x^{1/3}>0$, and $2-x>0$ (since $2>x$).

$$\Rightarrow \tilde{f}(x)>0 \Rightarrow f'(x)>0 \Rightarrow f \nearrow \text{ on } (0,2).$$

* If $2<x \Rightarrow 2-x<0$ and $x^{1/3}>0$

$$\Rightarrow \tilde{f}(x)<0 \Rightarrow \tilde{f}'(x)<0$$

$$\Rightarrow f \searrow \text{ on } (2,3) \text{ and on } (3,\infty) \text{ actually}$$

(g) Since $f \searrow$ on $(-\infty,0)$ and $f \nearrow$ on $(0,2)$.

$f(0)$ is a <u>local minimum</u>

Since $f \nearrow$ on $(0,2)$ and $f \searrow$ on $(2,3)$

$f(2)$ is a <u>local maximum</u>

Since $f \searrow$ on $(2,3)$ and $f \searrow$ on $(3,\infty)$ at there is <u>no</u> extremum $x=3$

since $f \xrightarrow[x \to -\infty]{} \infty$ and $f \xrightarrow[x \to +\infty]{} -\infty$,

those are just local extrema and there is no global extrema

(h) Concavity. We require to compute the 2nd derivative:

$$\frac{df(x)}{dx} = \frac{2(2-x)}{x^{1/3}(3-x)^{2/3}} = 2(2-x)x^{-1/3}(3-x)^{-2/3}$$

$$\frac{d^2 f}{dx^2} = 2\Big[(-1)x^{-1/3}(3-x)^{-2/3} + (2-x)\left(\frac{-1}{3}\right)x^{-4/3}(3-x)^{-2/3}$$

$$+ (2-x)x^{-1/3}\left(\frac{-2}{3}\right)(3-x)^{-5/3}(-1)\Big]$$

$$= \frac{-2x^{-4/3}(3-x)^{-5/3}}{3}\Big[3x(3-x) + (2-x)(3-x) - 2(2-x)x\Big]$$

$$= -\frac{2}{3x^{4/3}(3-x)^{5/3}}\Big[\cancel{9x} - \cancel{3x^2} + 6 - \cancel{5x} + \cancel{x^2} - \cancel{4x} + \cancel{2x^2}\Big]$$

$$= -2\frac{\cancel{(12x^2 - 12x + 6)}\ 6}{3x^{4/3}(3-x)^{5/3}}$$

$$\frac{d^2 f}{dx^2} = \frac{-12(2x^2 - 2x + 1)}{x^{4/3}(3-x)^{5/2}} = \frac{-4}{x^{4/3}(3-x)^{5/3}}$$

Observe that for the polynomial $2x^2 - 2x + 1$, we have discriminant $b^2 - 4ac = 4 - 4 \cdot 2 \cdot 1 = 4 - 8$
$= -4 < 0$.

Then, there is no roots and $2x^2 - 2x + 1 > 0$

Similarly, $2x^2 - 2x + 1 = 2\left[x^2 - x + \frac{1}{2}\right] =$

$$= 2\left[x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}\right] = 2\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right] > 0,$$

since result!

Then, for the 2nd derivative:

$$\frac{d^2f}{dx^2} = \frac{-12(2x^2 - 2x + 1)}{x^{4/3}(3-x)^{5/3}}$$

the portion $\dfrac{-12(2x^2-2x+1)}{x^{4/3}} < 0$, $\forall x \neq 0$.

Then, the sign of f'' is defined by $(3-x)^{5/3}$.

For $3 - x < 0$, i.e. $\boxed{x > 3,\ f'' > 0.}$

then $\boxed{f \text{ is concave up.}}$

For $3 - x > 0$, i.e. $\boxed{x < 3,\ f'' < 0}$ (with $x \neq 0$)

then $\boxed{f \text{ is concave down (except at: } x = 0)}$

(i) Then, f is continuous at $x = 3$,

and f changes concavity at $x = 3$

then $(3, f(3))$ is an inflection point

$f(3) = 2\sqrt[3]{3^2(3-3)} = 0$ i.e. at $(3, 0)$.

0606272020

(j) The only critical point s.t. $f'(x)=0$ is $\underline{x_1=2}$ | $f'(x_2)=0$

Now, $f''(2) = -\dfrac{12(2\cdot 2^2 - 2\cdot 2+1)}{2^{4/3}(3-2)^{5/2}} =$

$= \dfrac{-12(5)}{\sqrt[3]{16}\cdot 1} < 0.$

Then, $f(2)$ is a <u>local</u> <u>maximum</u> as we have claimed: $f(2) = 2\sqrt[3]{2^2(3-2)} = 2\cdot 2^{2/3} = 2^{5/3} = \sqrt[3]{32} \approx 3.2$

(k) So, we can know sketch the graph with the following information:

	$(-\infty,0)$	$(0,2)$	$(2,3)$	$(3,\infty)$
Monotony:	$f \searrow$	$\nearrow$	$\searrow$	$\searrow$.
Concavity	Concave down	Concave down	Concave down	Concave up
Extrema		$f(0)$ <u>loc min</u>	$f(2)$ <u>loc max</u>	<u>Nothing</u> In
Inflexion				Inflexion point at $x=3$

Also observe, $\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^-} \dfrac{2(2-x)}{x^{1/3}(3-x)^{2/3}} = \dfrac{4}{3^{2/3}} \lim\limits_{x\to 0^-} \dfrac{1}{x^{1/3}}$

$= -\infty$

Similarly

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2(2-x)}{x^{1/3}(3-x)^{2/3}} = \frac{4}{3^{2/3}} \lim_{x\to 0^+} \frac{1}{x^{1/3}}$$

$$= +\infty.$$

Then, we have a cusp and vertical tangency at $x=0$

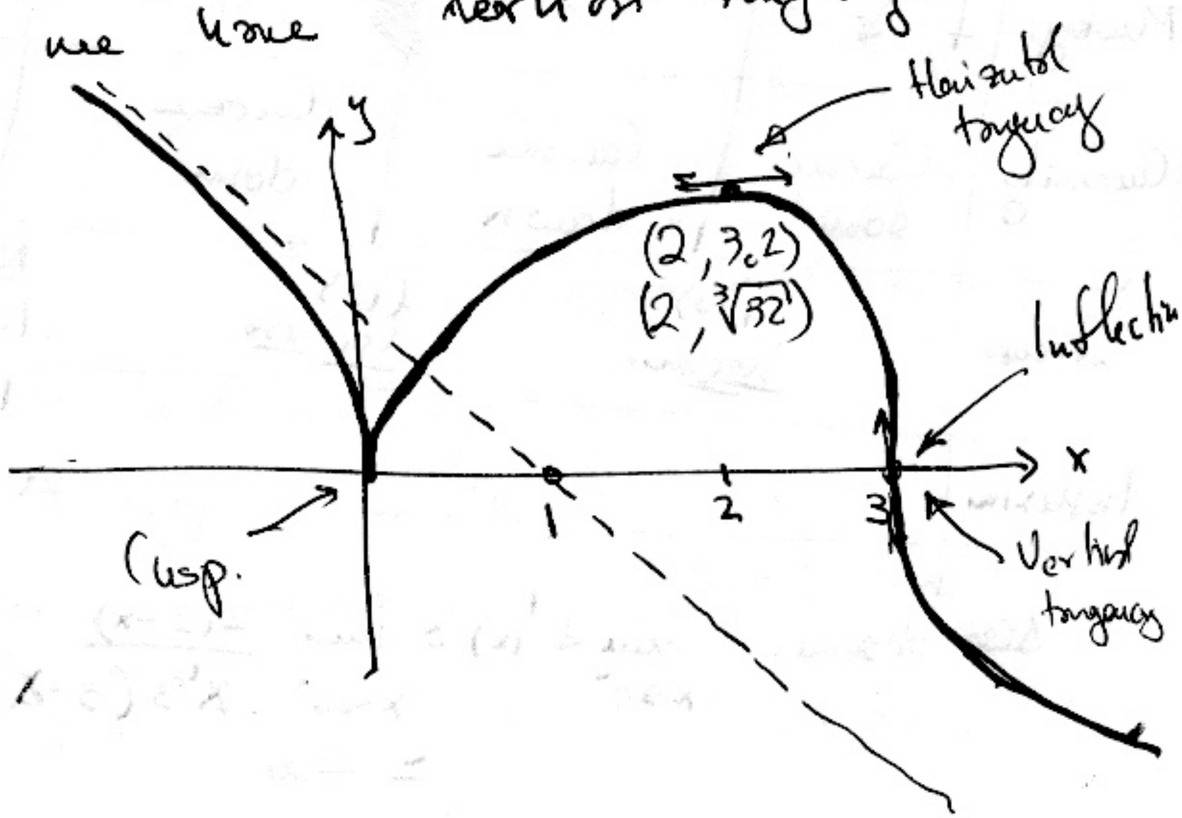
Now $\lim_{x\to 3} f'(x) = \lim_{x\to 3} \frac{2(2-\cancel{x})}{\cancel{x}^{1/3}(3-x)^{2/3}}$

$= -\frac{2}{3^{1/3}} \lim \frac{1}{(3-x)^{2/3}}$. Since we have $(3-x)^2$, have the same sign, then the limit is $+\infty$

and $\lim_{x\to 3} f'(x) = -\infty$

and we have vertical tangency.

(f)

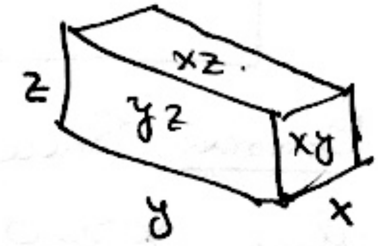


=8=

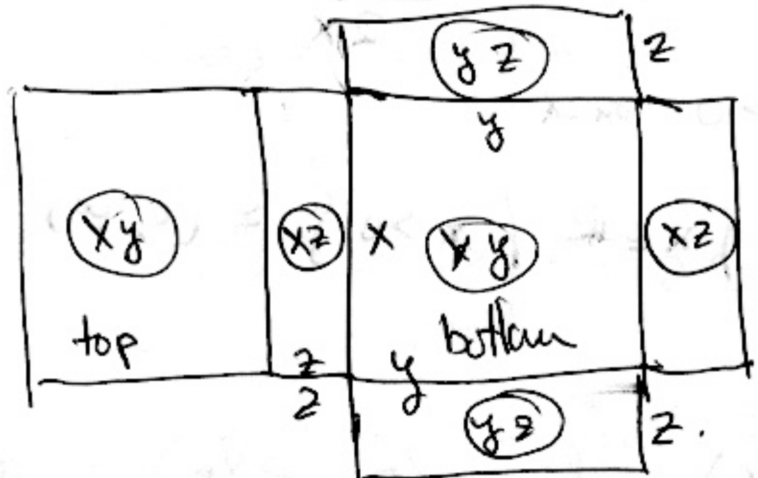
0606 272020

(2) Here, we have a box of sides x, y, z, with volume $V_0 = 10\ m^3$ and $V(x,y,z) = xyz = V_0$

The surface area of the box is.



$$A(x,y,z) = 2(xy + xz + yz)$$



The length "y" is twice the width "x":

$$y = 2x$$

Hence:

$$2x^2 z = V_0 \ldots\ldots (*)$$

Then, the area is: $\begin{cases} A(x,z) = 2(2x^2 + xz + 2xz) \\ A(x,z) = 2(2x^2 + 3xz) \end{cases}$

Now, from $(*)$: $z = \dfrac{V_0}{2x^2}$, we obtain $A = A(x)$

$$A(x) = 2\left(2x^2 + 3x\frac{V_0}{2x^2}\right) \Rightarrow \boxed{A(x) = 2\left(2x^2 + \frac{3V_0}{x}\right)}$$

This is the function to be minimized.

$$\mathrm{Dom}(A) = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty),$$

since x is a length, $x \geq 0$. Since it appears in denominator, $x \neq 0$: $\Longrightarrow x > 0$.

If the cost is $p\ \$/m^2$ $\Rightarrow$ the total cost: $\boxed{C(x) = p\,A(x)}$

$(p = 50)$.

Then, the function to be minimized is:

$$\boxed{C(x) = p\,A(x) = 2p\left(2x^2 + \frac{3V_0}{x}\right)} \text{ pesos}$$

Domain. Since the dimensions x, y, z are positive ≥ 0; then $x \geq 0$. But since: $\frac{3V_0}{2x}$ has a singularity at $x = 0$, then $x > 0$ and

$$Dom(C) = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty).$$

Observe that:

$$\lim_{x\to 0^+} C(x) = \lim_{x\to 0^+} 2p\left(\frac{3V_0}{2x}\right) = \infty.$$

and

$$\lim_{x\to \infty} C(x) = \lim_{x\to\infty} 2p(2x^2) = \infty$$

Then, it seems it should have at least a minimum value.

Now, we look for critical points:

$$C'(x) = 2p\left(4x - \frac{3V_0}{2x^2}\right).$$

There is no singularities on $(0, \infty)$, so $C'(x)$ always exist on $Dom(C)$.

Then, the only critical points are solutions to: 0606 27 2020.

$C'(x) = 0$, then $4x - \frac{3V_0}{2x^2} = 0 \Rightarrow x^3 = \frac{3V_0}{8}$

$\Rightarrow$ $\boxed{x_1 = \frac{1}{2}\sqrt[3]{3V_0}}$ is the only critical point.

Now, $C''(x) = 2p\left(4 + 2 \cdot \frac{3V_0}{2x^3}\right) = 2p\left(4 + \frac{3V_0}{x^3}\right)$

Then:

$$C''(x_1) = 2p\left(4 + 3V_0\left(\frac{8}{3V_0}\right)\right) = 2p(4+8) = 24p > 0$$

Then, $C(x_1)$ is a local minimum.

Actually, since there is only one critical point $x = x_1$, with $C'(x_1) = 0$, and observing that:

since $\forall x > 0$; $C''(x) = 2p\left(4 + \frac{3V_0}{x^3}\right) > 0$,

the graph of C is always concave up, then:

$C(x_1)$ should be a global minimum.

In particular the cost is:

$C(x_1) = 2p\left(2x_1^2 + \frac{3V_0}{x_1}\right) = \frac{p}{x_1}\left(4x_1^3 + 3V_0\right)$

$= \frac{p}{x_1}\left(4\left(\frac{3V_0}{8}\right) + 3V_0\right) = \frac{p}{x_1}\left(\frac{1}{2} + 1\right)3V_0 = \frac{9pV_0}{2x_1}$

$$C(x_1) = \frac{9V_0 p}{2x_1} \quad \text{i.e.} \quad C(x_1) = \frac{9}{2} V_0 p \, 2 \frac{1}{\sqrt[3]{3V_0}}$$

$$\Rightarrow \boxed{C(x_1) = \frac{9\, V_0^{2/3}}{\sqrt[3]{3}} p}$$

Since $V_0 = 10\,m^3$, $p = \$50/m^2$: $C(x_1) = \frac{9\,(10)^{2/3} \cdot 50}{\sqrt[3]{3}}$

$$\boxed{C(x_1) = 1{,}448.23 \text{ pesos}}$$

is the minimal cost. Quite expensive

Dimensions

$$x_1 = \frac{1}{2}\sqrt[3]{3V_0} \approx 1.5536\,m$$

$$y_1 = 2x_1 \approx 3.1072\,m$$

$$z_1 = \frac{V_0}{2x_1^2} \approx 2.0714\,m.$$

(3) (a) Enuncie el Teorema de Rolle.

Let f be a function with $Dom(f) = [a,b]$ a closed and bounded set. Let f be continuous on $[a,b]$, with continuous derivative on (a,b).

If $f(a) = f(b) = 0$, then, there exists a point $c \in (a,b)$ such that:

$$f'(c) = 0.$$

(b) Choose $[a,b]$ and f in easy way that satisfy Rolle's Thm hypotesis.

Two examples: Let $f(x) = 1 - x^2$ on $[-1, 1]$.

or $g(x) = \sin x$ on $[0, \pi]$.

Observe that f and g are continuous with

$f'(x) = -2x$ is continuous on $(-1,1)$

or $g'(x) = \cos x$ is continuous on $(0, \pi)$.

Notice: $f(-1) = f(1) = 1 - (\pm 1)^2 = 0$

$g(0) = g(2\pi) = \sin 0 = \sin(2\pi) = 0$.

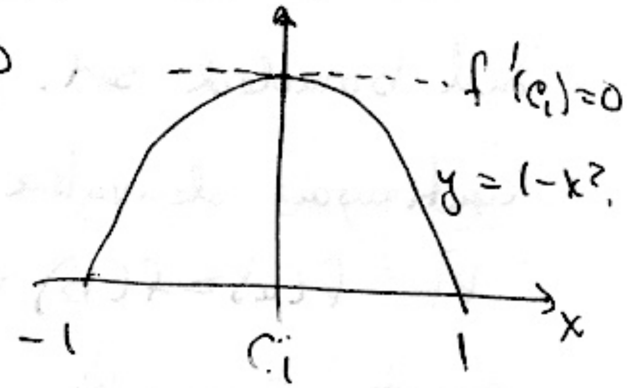
Then, there is a $c_1 \in [-1, 1]$, s.t $f'(c_1) = 0$

or $c_2 \in [0, \pi]$ s.t $g'(c_2) = 0$

(c) Find c for the given example.

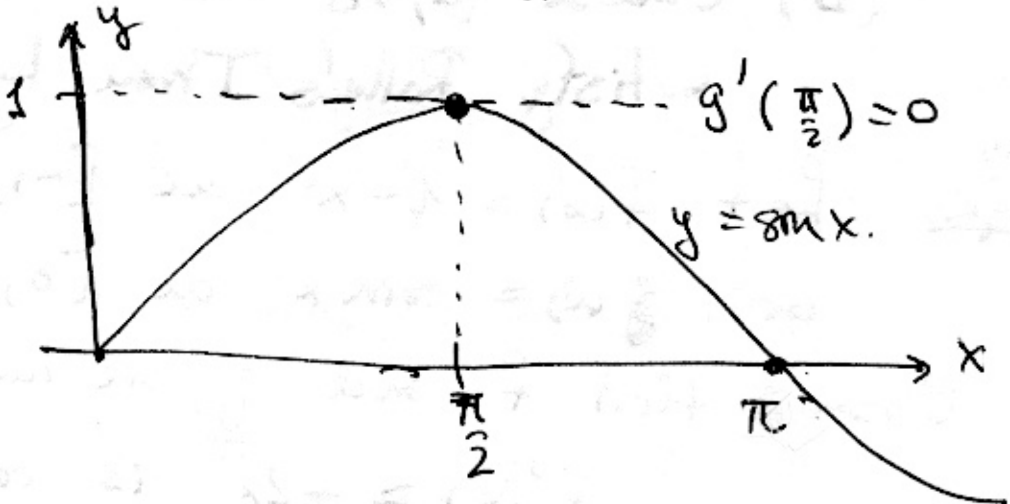
~~Here~~, $f'(c_1) = -2c_1 = 0$

then $c_1 = 0$



Also, $g'(c_2) = \cos(c_1) = 0$

Then, $c_1 = \frac{\pi}{2}$.



(4). A physical engineering wants to minimize the energy of a system. Say, the kinetic energy of a gas. depends on hundred thousand of particles.

$$K(x_1, x_2, \dots, x_n)$$

But if the particles interact among them with potential energy:

$$U(x_i, x_j) = U(|x_i - x_j|) = \text{Const}$$
$$(i = 1, 2, \dots, n)$$

then, we could reduce the function to:

$$K(x),$$

and employ the methods of Differential Calculus to solve it. by differentiating, finding the critical points and finding the minimus.

A chemical Engineering, which is interested in the pollution, want to know the amount of pollutant particles floating on air. He/she wants to know amount of winds and its velocities, and also the altitude over see level.

Then, if P is the concentration of a pollutant on air, v is the velocity of wind,

and if, h, is the altitude above sea level, then
$\dot{P}$ is a function of v and h.

$$P = P(v, h)$$

If we consider the sea level, $h = 0m$,
and then consider: $P = P(v)$.

Again, compute its derivative to find the
critical points, monotonicity, and then,
extrema of the pollution levels
