UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: OTOÑO DE 2020.
CÁLCULO INTEGRAL
EXAMEN # 1 (FORMA REMOTA). – A.
FECHA: LUNES 18 DE ENERO DE 2021.
HORA 16:00. HORA DE ENTREGA: 17:30 A 18:00

Nombre: ANSWER KEY

- El examen consta de **SEIS** problemas con diferentes puntajes.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL** y se resuelve de forma **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo. Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- **Para recibir puntaje**: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla con su examen escaneado.

(1) **(20 puntos)**

(a) Usando la definición, calcule $\int_{-1}^{2} x\, dx$

(b) Usando Geometría plana, calcule la misma integral.

(2) **(10 puntos)** Calcule la integral:

$$\int_{-\pi/4}^{\pi/4} \cos x + \frac{e^{-x^2}\log(1+x^2)\cos x}{\cot x}\, dx$$

(3) **(20 puntos)** Encuentre la antiderivada:

$$\int e^{-3x}\cos 3x\, dx.$$

(4) **(20 puntos)** Calcule la derivada de la siguiente función:

$$\int_{\cos x}^{\sin x} e^{y} y^2\, dy.$$

(5) **(20 puntos)** Calcule la integral:

$$\int_{0}^{\pi/4} 2\cos(2x)\sin(\sin(2x))\, dx.$$

(6) **(10 puntos)** Usando la definición, calcule la derivada de la siguiente función en el punto $x = 4$:

$$f(x) = \frac{1}{x}.$$

UAM-AZCAPOTZALCO | Lunes, 18 de enero de 2021

[CÁLCULO INTEGRAL] Examen #1-A Answer Key

① (a) We should use the Riemann sums to compute this integral. We divide the interval $[-1, 2]$, in N parts, and then choose the right-hand-side endpoints to evaluate the function $f(x) = x$.

The N subintervals have the same lengths:

$$\Delta x = \frac{2-(-1)}{N} = \frac{3}{N} = \frac{b-a}{N} > 0$$

Defining $x_0 = a = -1$, the k-th point in the partition is:

$$x_k = x_0 + k\Delta x. \quad , k = 1, 2, 3, \ldots, N.$$

Notice that $x_N = x_0 + N\left(\frac{b-a}{N}\right) = b$ ✓

The right endpoints are $x_1, x_2, \cdots, x_N$. Then, the Riemann sum is:

$$\sum_{k=1}^{N} f(x_k)\,\Delta x = \sum_{k=1}^{N} x_k\,\Delta x = \sum_{k=1}^{N}(x_0 + k\Delta x)\Delta x$$

$$= \sum_{k=1}^{N} x_0\Delta x + \sum_{k=1}^{N} k(\Delta x)^2 = x_0\Delta x\sum_{k=1}^{N} 1 + (\Delta x)^2\sum_{k=1}^{N} k$$

We could factor x_0 and Δx, since they are independent of k.

Now $\sum_{k=1}^{N} 1 = N$ and $\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$, by Gauss formula.

Then, the Riemann sum becomes:

$$\sum_{k=1}^{N} f(x_k)\,\Delta x = x_0\,\Delta x\,N + \Delta x^2\,\frac{N(N+1)}{2}$$

$$= x_0\,\frac{(b-a)}{N}\cdot N + \frac{(b-a)^2}{N^2}\,\frac{N(N+1)}{2}$$

$$= x_0\,(b-a) + \frac{(b-a)^2}{2}\,\frac{N}{N}\left(\frac{N+1}{N}\right)$$

Since $x_0 = a = -1$, $b = 2$: Then:

$$= (-1)(3) + \frac{3^2}{2}\left(1+\frac{1}{N}\right)$$

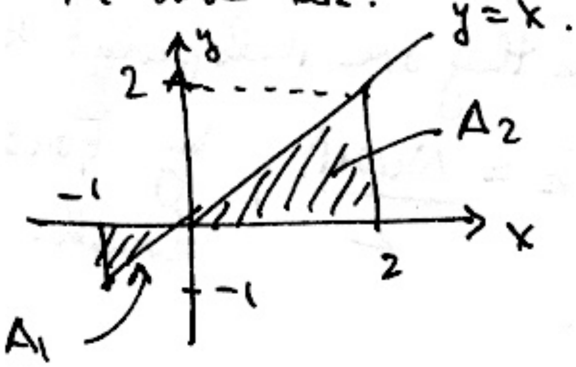
Then
$$\int_{-1}^{2} x\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x = \lim_{N\to\infty}\left(-3+\frac{3^2}{2}\left(1+\frac{1}{N}\right)\right)$$

$$= -3+\frac{3^2}{2} = \frac{-6+9}{2} = \frac{3}{2}$$

$$\boxed{\int_{-1}^{2} x\,dx = \frac{3}{2}}$$

(b) Using Geometry, we have two triangles of Areas A_1 and A_2:

$$A_1 = \frac{1\cdot 1}{2} = \frac{1}{2}\,,\quad A_2 = \frac{2\cdot 2}{2} = 2$$



$$\int_{-1}^{2} x\,dx = -A_1 + A_2 = -\frac{1}{2} + 2$$

$$= \frac{3}{2}$$ ✓ Same value!

(2) Observe that the interval of integration is symmetric with respect the origin; the first function, cos x, is and even function and the second function is an odd function because is the product of even functions times cot x, which is odd:

$$\int_{-\pi/4}^{\pi/4} \cos x + \frac{e^{-x^2}\operatorname{Log}(1+x^2)\cos x}{\cot x}\,dx =$$

Even function

Odd function

$$= \int_{-\pi/4}^{\pi/4} \cos x\,dx + \int_{-\pi/4}^{\pi/4} \frac{e^{-x^2}\operatorname{Log}(1+x^2)\cos x}{\cot x}\,dx$$

$$= 2\int_{0}^{\pi/4} \cos x\,dx + 0,$$ since it is odd, over a symmetric interval of integration

$$= 2 \sin x \Big|_{0}^{\pi/4} + 0$$

$$= 2\sin\left(\frac{\pi}{4}\right) = 2\,\frac{\sqrt{2}}{2} = \sqrt{2}$$

(3) This is a cyclic integral. We have to integrate twice by parts. However, to simplify the computation, we will first perform a change of variables $y = -3x$:

$$\int e^{-3x} \cos 3x \, dx = -\frac{1}{3} \int e^{y} \cos y \, dy =$$

$$= -\frac{1}{3} \left[e^{y} \cos y - \int e^{y} (-\sin y) \, dy \right]$$ by integration by parts

$$= -\frac{1}{3} \left[e^{y} \cos y + \int e^{y} \sin y \, dy \right]$$

$$= -\frac{1}{3} \left[e^{y} \cos y + e^{y} \sin y - \int e^{y} \cos y \, dy \right]$$

$$= -\frac{1}{3} \left(e^{y} \cos y + e^{y} \sin y \right) - \left(-\frac{1}{3}\right) \int e^{y} \cos y \, dy$$

Returning to the original variables:

$$= -\frac{1}{3} \left[e^{-3x} \cos 3x - e^{-3x} \sin 3x \right] - \int e^{-3x} \cos 3x \, dx$$

Then

$$2 \int e^{-3x} \cos 3x \, dx = -\frac{1}{3} e^{-3x} \left[\cos 3x - \sin 3x \right] + C_1$$

$$\Rightarrow \boxed{\int e^{-3x} \cos 3x \, dx = \frac{1}{6} e^{-3x} \left[\sin 3x - \cos 3x \right] + C_2}$$

④ Compute the derivative of the function $\int_{\cos x}^{\sin x} e^y y^2\,dy$

There are two ways:

<u>way ①</u> <u>Easy way</u> Using the Fundamental Theorem of Calculus, and integrating by parts twice, and using Chain rule.

$$\frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^y y^2\,dy\right) = \frac{d}{dx}\left(\int_{\cos x}^{0} e^y y^2\,dy + \int_{0}^{\sin x} e^y y^2\,dy\right),$$ by properties of integrals.

$$= \frac{d}{dx}\left(-\int_{0}^{\cos x} e^y y^2\,dy + \int_{0}^{\sin x} e^y y^2\,dy\right),$$ by properties of the integrals.

Define $u(x) = \cos x$, $v(x) = \sin x$.

$$= -\frac{d}{dx}\left(\int_0^{u(x)} e^y y^2\,dy\right) + \frac{d}{dx}\left(\int_0^{v(x)} e^y y^2\,dy\right),$$ by properties of derivatives

By the Chain rule:

$$= -\frac{d}{du}\left(\int_0^{u} e^y y^2\,dy\right)\frac{du}{dx} + \frac{d}{dv}\left(\int_0^{v} e^y y^2\,dy\right)\frac{dv}{dx}$$

By the Fundamental Thm of Calculus

$$= -(e^u u^2)\frac{du}{dx} + e^v v^2 \frac{dv}{dx}.$$

Finally, substitute $u = \cos x$, $v = \sin x$;

$$= e^{\cos x}\cos^2 x \sin x + e^{\sin x}\sin^2 x \cos x.$$

$$= \cos x \sin x\left(e^{\cos x}\cos x + e^{\sin x}\sin x\right).$$

Way ② This is the long way. You have to integrate by parts twice, evaluate at the integral limits, then compute the derivative: Compute the antiderivative:

$$\int e^y y^2\,dy = e^y y^2 - \int e^y\, 2y\,dy \quad \text{, by parts}$$

$$= e^y y^2 - 2\int e^y y\,dy$$

$$= e^y y^2 - 2\left[e^y y - \int e^y\,dy\right] \quad \text{by parts again}$$

$$= e^y y^2 - 2e^y y + 2e^y + C.$$

Evaluate at $\cos x$ and $\sin x$:

$$= e^y(y^2 - 2y + 2) + C.$$

$$\int_{\cos x}^{\sin x} e^y y^2\,dy = e^{\sin x}(\sin^2 x - 2\sin x + 2) - e^{\cos x}(\cos^2 x - 2\cos x + 2)$$

Then, compute the derivatives.

$$\frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^y y^2\,dy\right) = \cos x\, e^{\sin x}(\sin^2 x \cancel{- 2\sin x + 2}).$$
$$+ e^{\sin x}(\cancel{2\sin x\cos x - 2\cos x}) +$$
$$+ \sin x\, e^{\cos x}(\cos^2 x \cancel{- 2\cos x + 2})$$
$$+ e^{\cos x}(\cancel{2\cos x(\sin x) - 2(\sin x)}).$$

$$= \cos x\, e^{\sin x}\sin^2 x + \sin x\, e^{\cos x}\cos^2 x. \quad \text{Same result!}$$

Remark I have skipped some steps. Please fill them in.

⑤ Compute the following integral. It is a usual problem of Substitution

$$\int_0^{\pi/4} 2\cos(2x)\sin(\sin(2x))\,dx = \int_0^{\pi/4} \frac{d}{dx}(\sin 2x)\,\sin(\sin(2x))\,dx$$

Observe we have a derivative in the integrand.

Then choose:

$u = \sin 2x$

<u>Limits of integration</u>: $x_1 = 0 \Rightarrow u_1 = \sin 2x_1 = \sin 0 = 0$

$x_2 = \frac{\pi}{4} \Rightarrow u_2 = \sin(2x_2) = \sin\frac{\pi}{2} = 1$

Then:

$$= \int_0^1 \sin(u)\,du = -\cos u \Big|_0^1 = -\cos(1) + \cos(0)$$

Therefore:

$$\boxed{\int_0^{\pi/4} 2\cos(2x)\sin(\sin(2x))\,dx = 1 - \cos(1)}$$

⑥ This is a present from you, from Diff. Calculus

We need to compute:

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{4 - (4+h)}{(4+h)4} \right) = \lim_{h \to 0} \frac{1}{h} \left(\frac{-h}{(4+h)4} \right)$$

$$= \lim_{h \to 0} \frac{-1}{(4+h)4} = \frac{-1}{(4)(4)}$$

$$\boxed{f'(4) = -\frac{1}{16}.}$$