UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: OTOÑO DE 2020.
CÁLCULO INTEGRAL
EXAMEN # 1 (FORMA REMOTA). — B.
FECHA: LUNES 18 DE ENERO DE 2021.
HORA 16:00. HORA DE ENTREGA: 17:30 A 18:00

Nombre: ______________________________

- El examen consta de **SEIS** problemas con diferentes puntajes.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL** y se resuelve de forma **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo. Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- **Para recibir puntaje**: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla con su examen escaneado.

(1) **(20 puntos)**

(a) Usando la definición, calcule $\int_{-1}^{2} x\,dx$

(b) Usando Geometría plana, calcule la misma integral.

(2) **(10 puntos)** Calcule la integral:

$$\int_{-\pi/4}^{\pi/4} \cos x + \frac{e^{-x^2}\log(1+x^2)\cos x}{\cot x}\,dx$$

(3) **(20 puntos)** Encuentre la antiderivada:

$$\int e^{2x}\sin 2x\,dx.$$

(4) **(20 puntos)** Calcule la derivada de la siguiente función:

$$\int_{x^2}^{x^3} e^{y}\cos y\,dy.$$

(5) **(20 puntos)** Calcule la integral:

$$\int_{0}^{3\pi/2} \frac{1}{3}\cos\left(\frac{x}{3}\right)\sin\left(\sin\left(\frac{x}{3}\right)\right)\,dx.$$

(6) **(10 puntos)** Usando la definición, calcule la derivada de la siguiente función en el punto $x = 4$.

$$f(x) = \sqrt{x}.$$

CÁLCULO INTEGRAL | Examen #1-B ANSWER KEY

(1) (a) We should use the Riemann sums to compute this integral. We divide the interval $[-1, 2]$ in N parts, and then choose the right-hand-side endpoints to evaluate the function $f(x) = x$.

The N subintervals have the same lengths:

$$\Delta x = \frac{2-(-1)}{N} = \frac{3}{N} = \frac{b-a}{N} = b$$

Defining $x_0 = a = -1$, the k-th point in the partition is:

$$x_k = x_0 + k\Delta x, \quad k = 1, 2, 3, \ldots, N.$$

Notice that $x_N = x_0 + N\left(\frac{b-a}{N}\right) = b$ ✓

The right endpoints are $x_1, x_2, \ldots, x_N$. Then, the Riemann sum is:

$$\sum_{k=1}^{N} f(x_k)\,\Delta x = \sum_{k=1}^{N} x_k\,\Delta x = \sum_{k=1}^{N}(x_0 + k\Delta x)\Delta x$$

$$= \sum_{k=1}^{N} x_0\Delta x + \sum_{k=1}^{N} k(\Delta x)^2 = x_0\Delta x\sum_{k=1}^{N} 1 + (\Delta x)^2\sum_{k=1}^{N} k$$

We could factor x_0 and Δx, since they are independent of k.

Now $\sum_{k=1}^{N} 1 = N$ and $\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$, by Gauss formula.

Then, the Riemann sum becomes:

$$\sum_{k=1}^{N} f(x_k)\,\Delta x = x_0\,\Delta x\,N + \Delta x^2\,\frac{N(N+1)}{2}$$

$$= x_0\,\frac{(b-a)}{N}\cdot N + \frac{(b-a)^2}{N^2}\,\frac{N(N+1)}{2}$$

$$= x_0\,(b-a) + \frac{(b-a)^2}{2}\,\frac{N}{N}\left(\frac{N+1}{N}\right)$$

Since $x_0 = a = -1$, $b = 2$: Then:

$$= (-1)(3) + \frac{3^2}{2}\left(1+\frac{1}{N}\right)$$

Then

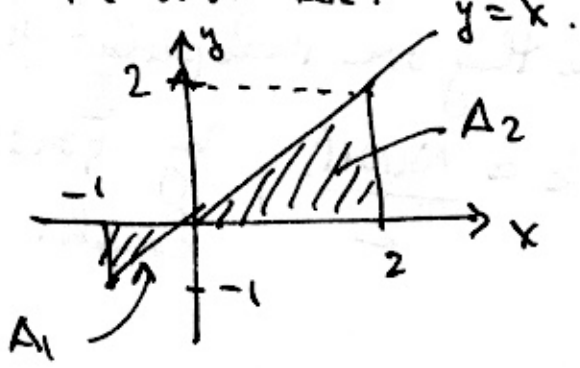
$$\int_{-1}^{2} x\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x = \lim_{N\to\infty}\left(-3+\frac{3^2}{2}\left(1+\frac{1}{N}\right)\right)$$

$$= -3 + \frac{3^2}{2} = \frac{-6+9}{2} = \frac{3}{2}$$

$$\boxed{\int_{-1}^{2} x\,dx = \frac{3}{2}}$$

(b) Using Geometry, we have two triangles of Areas A_1 and A_2:

$$A_1 = \frac{1\cdot 1}{2} = \frac{1}{2}, \quad A_2 = \frac{2\cdot 2}{2} = 2$$



$$\int_{-1}^{2} x\,dx = -A_1 + A_2 = -\frac{1}{2} + 2$$

$$= \frac{3}{2}$$ ✓ Same value!

(2) Observe that the interval of integration is symmetric with respect the origin; the first function, $\cos x$, is and even function and the second function is an odd function because is the product of even functions times $\cot x$, which is odd:

$$\int_{-\pi/4}^{\pi/4} \underbrace{\cos x}_{\text{Even function}} + \underbrace{\frac{e^{-x^2}\operatorname{Log}(1+x^2)\cos x}{\cot x}}_{\text{Odd function}}\, dx =$$

$$= \int_{-\pi/4}^{\pi/4} \cos x \, dx + \int_{-\pi/4}^{\pi/4} \frac{e^{-x^2}\operatorname{Log}(1+x^2)\cos x}{\cot x}\, dx$$

$$= 2\int_{0}^{\pi/4} \cos x \, dx + 0,$$ since it is odd, over a symmetric interval of integration

$$= 2 \sin x \Big|_0^{\pi/4} + 0$$

$$= 2 \sin\left(\frac{\pi}{4}\right) = 2\,\frac{\sqrt{2}}{2} = \boxed{\sqrt{2}}$$

③ This is a cyclic integral: We have to integrate by parts twice. However, to simplify the computation, we will first perform the change of variables $y = 2x$.

$$\int e^{2x} \sin 2x \, dx = \frac{1}{2}\int e^{y} \sin y \, dy$$

$$= \frac{1}{2}\left[e^{y} \sin y - \int e^{y} \cos y \, dy \right], \text{ by integration by parts.}$$

$$= \frac{1}{2}\left[e^{y} \sin y - \left(e^{y} \cos y - \int e^{y} (-\sin y)\, dy \right)\right]$$

by a 2nd integration by parts

$$= \frac{1}{2}\left[e^{y}(\sin y - \cos y) - \int e^{y} \sin y \, dy \right]$$

$$= \frac{1}{2}\left[e^{y} (\sin y - \cos y)\right] - \frac{1}{2}\int e^{y} \sin y \, dy$$

Returning to the original variable

$$= \frac{1}{2} e^{2x}(\sin 2x - \cos 2x) - \underbrace{\int e^{2x} \sin 2x \, dx}_{\text{It is on the left-hand side}}$$

Then:

$$2 \int e^{2x} \sin 2x \, dx = \frac{1}{2} e^{2x} (\sin 2x - \cos 2x) + C_1.$$

Hence $\boxed{\int e^{2x} \sin 2x \, dx = \frac{1}{4} e^{2x}\left(\sin 2x - \cos 2x\right) + C_2}$

(1) There are two ways to solve this problem:

Way (i) Use the Fundamental Thm of Calculus, and properties of integrals, and the Chain rule:

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} e^y \cos y \, dy\right) = \frac{d}{dx}\left(\int_{x^2}^{0} e^y \cos y \, dy + \int_{0}^{x^3} e^y \cos y \, dy\right)$$

Properties of integrals

$$= \frac{d}{dx}\left(-\int_{0}^{x^2} e^y \cos y \, dy + \int_{0}^{x^3} e^y \cos y \, dy\right)$$

Now, if we define $u(x) = x^2$, and $v(x) = x^3$

$$= -\frac{d}{dx}\left(\int_{0}^{u(x)} e^y \cos y \, dy\right) + \frac{d}{dx}\left(\int_{0}^{v(x)} e^y \cos y \, dy\right)$$ by properties of derivatives

and by the Chain rule.

$$= -\frac{d}{du}\left(\int_{0}^{u} e^y \cos y \, dy\right)\frac{du}{dx} + \frac{d}{dv}\left(\int_{0}^{v} e^y \cos y \, dy\right)\frac{dv}{dx}.$$

By the Fundamental Theorem of Calculus:

$$= -\left(e^u \cos u\right)\frac{du}{dx} + \left(e^v \cos v\right)\frac{dv}{dx}.$$

and substituting $u = x^2$, $v = x^3$:

$$= -2x\, e^{x^2} \cos(x^2) + 3x^2\, e^{x^3} \cos x^3$$

<u>Long way</u> <u>Way ②</u> This is the long way. We have to find the antiderivative:

$$\int e^y \cos y \, dy = e^y \cos y + \int e^y \sin y \, dy \quad \left(\text{by parts twice}\right)$$

$$= e^y \cos y + e^y \sin y - \int e^y \cos y \, dy$$

then.

$$\int e^y \cos y \, dy = \frac{1}{2} e^y \left(\cos y + \sin y\right) + C.$$

(Same as in problem ③).. Evaluate at x^2 and x^3, and subtract:

$$\int_{x^2}^{x^3} e^y \cos y \, dy = \frac{1}{2} e^{x^3}\left(\cos x^3 + \sin x^3\right) - \frac{1}{2} e^{x^2}\left(\cos x^2 + \sin x^2\right)$$

and compute the derivative:

$$\frac{d}{dx}\int_{x^2}^{x^3} e^y \cos y \, dy = \frac{1}{2} 3x^2 e^{x^3}\left[\left(\cos x^3 + \sin x^3\right) + \left(-\sin x^3 + \cos x^3\right)\right]$$

$$- \frac{1}{2} 2x^2 e^{x^2}\left[\left(\cos x^2 + \sin x^2\right) + \left(-\sin x^2 + \cos x^2\right)\right]$$

$$= 3x^2 e^{x^3} \cos x^3 - x^2 e^{x^2} \cos x^2.$$

<u>Same result!</u>

<u>Note</u> I skipped many steps when computing the indefinite integrals and derivatives.
Please fill them in.

⑤ This is a usual problem of change of variables:

$$\int_0^{3\pi/2} \frac{1}{3}\cos\left(\frac{x}{3}\right) \sin\left(\sin\left(\frac{x}{3}\right)\right) dx$$

Observe that $\frac{1}{3}\cos\left(\frac{x}{3}\right) = \frac{d}{dx}\left(\sin\left(\frac{x}{3}\right)\right)$ in the integrand.

Then, the change of variables is $u = \sin\left(\frac{x}{3}\right)$.

Now $x_1 = 0 \Rightarrow u_1 = \sin\left(\frac{0}{3}\right) = 0$

$x_2 = \frac{3\pi}{2} \Rightarrow u_2 = \sin\left(\frac{1}{3}\frac{3\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

Hence,

$$\int_{x_1}^{x_2} \frac{d}{dx}(u(x)) \sin(u(x))\, dx = \int_{u_1}^{u_2} \sin(u)\, du = \int_0^1 \sin u\, du$$

$$= -\cos u \Big|_0^1 = -\cos(1) + \cos 0 = 1 - \cos(1)$$

$$\boxed{\int_0^{3\pi/2} \frac{1}{3}\cos\left(\frac{x}{3}\right) \sin\left(\sin\left(\frac{x}{3}\right)\right) dx = 1 - \cos(1)}$$

(6) This is a gift for you from Differential Calculus;

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{4+h} - 2)}{h} \frac{(\sqrt{4+h} + 2)}{(\sqrt{4+h} + 2)}$$

$$= \lim_{h\to 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4}.$$

Then $\boxed{f'(4) = \frac{1}{4}}$.