UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: OTOÑO DE 2020.
ECUACIONES DIFERENCIALES ORDINARIAS.
EXAMEN # 1 (FORMA REMOTA). A
FECHA: VIERNES 22 DE ENERO DE 2021.
HORA 17:30. HORA DE ENTREGA: 19:00 A 19:30

Nombre: ANSWER KEY.

- El examen consta de **CINCO** problemas de 20 puntos cada uno.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL** y se resuelve de forma **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo. Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- **Para recibir puntaje**: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla con su examen escaneado.

(1) **(20 puntos)** ¿De cuáles de las siguientes ecuaciones diferenciales es solución la siguiente función? (Nota: puede ser más de una ecuación.) $\phi(t) = \dfrac{1}{(t-1)^2}$

(a) $\dfrac{dy}{dt} = -2\dfrac{y}{t-1}$.

(b) $\dfrac{dy}{dt} = \dfrac{y}{(t-1)^2}$.

(c) $\dfrac{dy}{dt} = -2y^{3/2}$.

(2) **(20 puntos)**

(a) Escriba el esquema de Euler para el problema de valores iniciales siguiente, con paso arbitrario Δt,

$$\begin{aligned}\frac{dy}{dt} &= t^2 + y^2;\\ y(0) &= 1\end{aligned}$$

(b) Si $\Delta t = 1/10$, calcule los primeros valores aproximados para $y(1/10), y(2/10), y(3/10)$.

(3) **(20 puntos)** Dibuje la linea fase y bosqueje algunas soluciones representativas de la siguiente ecuación diferencial. También clasifique los puntos fijos geométricamente y usando el teorema de linealización. $\dfrac{dy}{dt} = y^3 + y^2 - 20y$.

(4) **(20 puntos)** Resuelva la ecuación diferencial: $3\dfrac{dy}{dt} + 9y = 6t - 15$, usando

(a) el método de coeficientes indeterminados.

(b) el método de factor integrante.

(5) **(20 puntos)** Usando la definición, calcule la derivada de la siguiente función en el punto $x = 4$:

$$f(x) = \frac{1}{x}.$$

(1) We have the function $\phi(t) = \frac{1}{(t-1)^2}$, whose derivative is: $\phi' = -\frac{2}{(t-1)}$

The right-hand-sides of the equations are

(a): $-2\frac{\phi}{t-1} = -\frac{2}{(t-1)}\frac{1}{(t-1)^2} = -\frac{2}{(t-1)^3}$ ✓ It is solution!

(b) $\frac{\phi}{(t-1)^2} = \frac{1}{(t-1)^2(t-1)^2} = \frac{1}{(t-1)^4}$ It is not a solution.

(c) $-2\phi^{3/2} = -2\left(\frac{1}{t-1}\right)^{3/2} = -2\frac{1}{(t-1)^3}$ ✓ It is solution!

Then, $\phi(t)$ solves eq'n's (a) and (c).

Moraleja: A function could be a solution of more than one equation.

(2)(a) The time step Δt allows to compute the time to compute: $t_k = t_0 + k\Delta t$, $k = 0, 1, 2, \ldots N$
(or $t_{k+1} = t_k + \Delta t$).
where $N \in \mathbb{N}$ (natural number). Here, $t_0 = 0$.
Euler's method to approximate $y(t_k)$ is:

$$y_{k+1} = f(t_k, y_k)\Delta t + y_k, \quad k = 0, 1, 2, \ldots, N$$

for the Diff Eq: $\quad \frac{dy}{dt} = f(t, y)$, with $y(0) = y_0$

Then, Euler scheme is:

$$\boxed{\begin{aligned} y_{k+1} &= (t_k^2 + y_k^2)\Delta t + y_k \\ y_0 &= 1. \\ t_k &= k\Delta t \end{aligned}}$$

$t_0 = 0$

$k = 0, 1, 2, \ldots N.$

(b) Now, for $\Delta t = 1/10$:

$$t_1 = \frac{1}{10}, \quad t_2 = \frac{2}{10}, \quad t_3 = \frac{3}{10}.$$

$$y_1 = (t_0^2 + y_0^2)\Delta t + y_0 = (0^2 + 1^2)\frac{1}{10} + 1 = \underline{1.1}$$

$$y_2 = (t_1^2 + y_1^2)\Delta t + y_1 = \left(\frac{1}{10}^2 + (1.1)^2\right)\frac{1}{10} + 1.1 = \underline{1.222}$$

$$y_3 = (t_2^2 + y_2^2)\Delta t + y_2 = \left(\left(\frac{2}{10}\right)^2 + (1.222)^2\right)\frac{1}{10} + 1.222$$

$$= \underline{\underline{1.3753284}}$$

The Differential Equation $\frac{dy}{dt} = y^3 + y^2 - 20y$ has stationary solutions (since it is autonomous) which satisfy the algebraic eq'n $y^3 + y^2 - 20y = 0$

i.e. $y(y^2+y-20)=0$

i.e. $y(y+5)(y-4)=0 \Rightarrow$

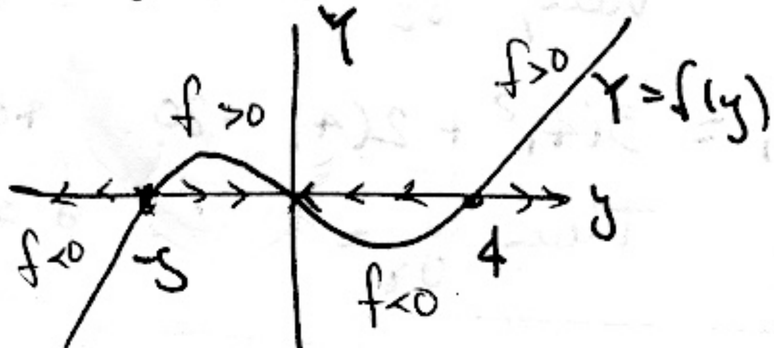
Stationary solutions

$y_1(t) = -5 = y_1$

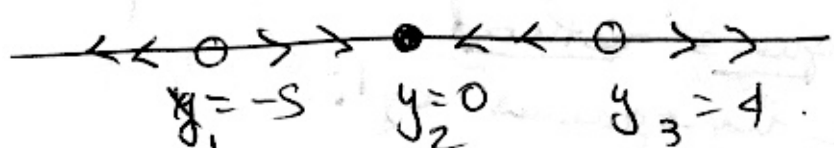
$y_2(t) = 0 = y_2$

$y_3(t) = 4 = y_3$.

The graph of $f(y) = y^3+y^2-20y$ looks like:

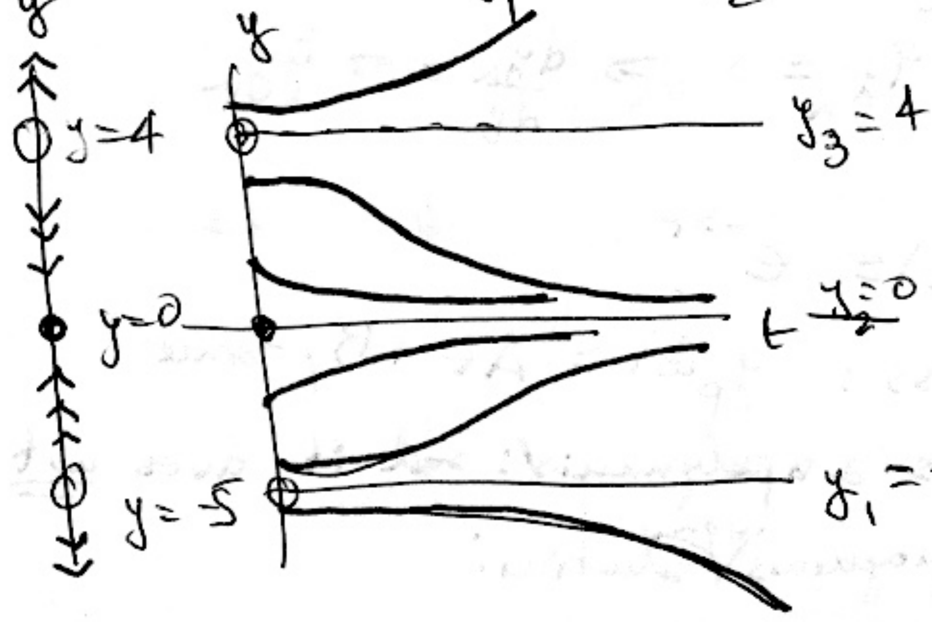


Then, the phase line is



$y_1 = -5$ is a source

$y_2 = 0$ is a sink



$y_3 = 4$ is a source

According with the linearization theorem, we req.

$f'(y) = \frac{df}{dy} = 3y^2 + 2y - 20$:

$f'(y_1) = 3(-5)^2 + (2)(-5) - 20 = 75 - 10 - 20 = 45 > 0$

Then $y_1 = -5$ is a <u>source</u>

$f'(y_2) = 3\cdot 0 + 2\cdot 0 - 20 = -20 < 0$

Then, $y_2 = 0$ is a <u>sink</u>

$f'(y_3) = 3(4)^2 + 2(4) - 20 = 48 + 8 - 20 = 36 > 0$

then $y_3 = 4$ is a <u>source</u>

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(4) $\cdot\ 3\frac{dy}{dt} + 9y = 6t - 15$

1) Linear equation
2) Const. Coeff's

(a) <u>Lucky guess method</u>..

(i) Solve homogeneous eq'n:

$$3\frac{dy_h}{dt} + 9y_h = 0 \Rightarrow \frac{dy_h}{dt} = -3y_h$$

$$\Rightarrow y_h(t) = e^{-3t}$$

(ii) Lucky guess: $y_p(t) = At + B$, since $b(t) = 6t - 15$ is a polynomial: and it does <u>not</u> repeat the homogeneous eqn solution:

$$3\frac{dy_p}{dt} + 9y_p = 6t - 15.$$

$$3(A) + 9(At + B) = 6t - 15.$$

$$9At + (3A + 9B) = 6t - 15$$

$9A = 6. \Rightarrow \boxed{A = 2/3}$

$3A + 9B = -15 \Rightarrow A + 3B = -5 \Rightarrow \frac{2}{3} + 3B = -5$

$\Rightarrow 3B = -5 - \frac{2}{3} = \frac{-15-2}{3} \Rightarrow \boxed{B = \frac{-17}{9}}$

By Principles $\Rightarrow$ of Linearity $\boxed{y(t) = Ce^{-3t} + \left(\frac{2}{3}t - \frac{17}{9}\right)}$

is the solution to the Diff Eq.

(b) Integrating factor method. We should take the equation to standard form: $1 \cdot \frac{dy}{dt} + 3y = 2t - 5.$

Integrating factor: $\mu(t) = e^{\int 3\,dt} = e^{3t}.$

(Remember $\mu(t) = y_h^{-1}(t) = (e^{-3t})^{-1} = e^{3t}$, coincides)

We can use the formula:

$$y(t) = \frac{1}{\mu(t)}\int b(t)\mu(t)\,dt + \frac{C}{\mu(t)}$$

We have to compute the integral:

=5=

$$\int b(t)\,\mu(t)\,dt = \int (2t-5)e^{3t}\,dt = 2\int te^{3t}dt - 5\int e^{3t}$$

By parts

$$= 2\left[\frac{te^{3t}}{(3)} - \frac{1}{3}\int e^{3t}dt\right] - 5\,\frac{e^{3t}}{3}$$

$$= 2\left[\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}\right] - \frac{5}{3}e^{3t}$$

$$= \left(\frac{2}{3}t - \frac{2}{9} - \frac{5}{3}\right)e^{3t} = \left(\frac{2}{3}t + \frac{-2-15}{9}\right)e^{3t}$$

$$= \left(\frac{2}{3}t - \frac{17}{9}\right)e^{3t}.$$

Then, the solution is.

$$y(t) = \frac{1}{\mu(t)}\int b(t)\,\mu(t)\,dt + \frac{c}{\mu(t)}$$

$$= \frac{1}{e^{3t}}\left(\frac{2}{3}t - \frac{17}{9}\right)e^{3t} + \frac{c}{e^{3t}}$$

$$= \boxed{\left(\frac{2}{3}t - \frac{17}{9}\right)e + Ce^{-3t}}$$

Same answer.

5) This is a present from Differential Calculus, but we all now see this definition is fundamental in Science and Engineering.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h\to 0} \frac{\frac{1}{h+x} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{x-(x+h)}{(x+h)x}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(-h)}{(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}} \quad \text{at } x=4 \quad \boxed{\left.\frac{d}{dx}\left(\frac{1}{x}\right)\right|_{x=4} = \frac{-1}{16}}$$