UAM - AZCAPOTZALCO | CÁLCULO DIFERENCIAL | Martes, Septiembre 28, 2021

Examen #2: ANSWER KEY

① (a) Domain of f. f is not defined if. $8-2x^2=0$

i.e. $4-x^2=0 \Rightarrow (2-x)(2+x)=0 \Rightarrow x=-2$, $x=+2$

Then:

$$\text{Dom}(f)=\{x\in\mathbb{R} \mid x\neq\pm 2\}=(-\infty,-2)\cup(-2,2)\cup(2,\infty)$$

(b) Axes - intercepts.

$f(x)=0 \Rightarrow \dfrac{x^2}{8-4x^2}=0 \Rightarrow x=0$ is the only x-intercept

$f(0)=0, \Rightarrow y=0$ is the only one y-intercept

It intersects the point (0,0) only

(c) Observe that: $f(-x)=\dfrac{(-x)^2}{8-2(-x)^2}=\dfrac{x^2}{8-2x^2}=f(x)$

i.e. f is even. We only have to study on $[0,\infty)$

(d) We should compute:

$$\lim_{x\to\infty}\frac{x^2}{8-2x^2}=\lim_{x\to\infty}\left(\frac{x^2}{x^2}\,\frac{1}{\left(\frac{8}{x^2}-2\right)}\right)=$$

$$=\lim_{x\to\infty}\frac{1}{\frac{8}{x^2}-2}=\frac{1}{0-2}=-\frac{1}{2}$$

Then, $\boxed{y=-1/2 \text{ is a horizontal asymptote}}$

(c) We need to compute f' to get the critical points:

$$\frac{df}{dx} = \frac{d}{dx}\frac{x^2}{2(4-x^2)} = \frac{1}{2}\frac{d}{dx}\frac{x^2}{4-x^2} = \frac{1}{2}\frac{(4-x^2)2x - x^2(-2x)}{(4-x^2)^2}$$

$$= \frac{1}{2}\frac{8x - 2x^3 + 2x^3}{(4-x^2)^2} = \frac{1}{2}\frac{8x}{(4-x^2)^2}$$

i.e. $\frac{df}{dx} = \frac{4x}{(4-x^2)^2}$

Now: (a) $\frac{df}{dx}(0) = 0$ if $\frac{4x}{(4-x^2)^2} = 0 \Rightarrow 4x = 0$

$\Rightarrow x = 0$

(b) $\frac{df}{dx}$ does not exist, if $4 - x^2 = 0 \Rightarrow \begin{cases} x = 2 \\ x = -2 \end{cases}$

but $2, -2 \notin \mathrm{Dom}(f)$. Then, they are no **not** critical points.

(c) There are **no** boundaries in $\mathrm{Dom}(f)$:

Then, $\boxed{x=0}$ is the only critical point.

(d) Again, in $x = 2$, we have a vertical asymptote.

$$\lim_{x\to 2^-}\frac{x^2}{2(4-x^2)} = \lim_{x\to 2^-}\frac{x^2}{2(2-x)(2+x)} = \frac{2^2}{2(2+2)}\lim_{x\to 2^-}\frac{1}{x-2}$$

$$= \frac{1}{2}(-\infty) = -\infty$$

Similarly $\lim_{x\to 2^+}\frac{x^2}{2(4-x^2)} = \frac{2^2}{2(2+2)}\lim_{x\to 2^+}\frac{1}{(x-2)} = +\infty$

$=2=$

(f) Monotonicity.

Notice that $\frac{df}{dx} = \frac{4x}{(4-x^2)} > 0$, for $x > 0$.

Then, $f \nearrow$ in $(0,2)$ and in $(2,\infty)$.

Similarly, $\frac{df}{dx} = \frac{4x}{4-x^2} < 0$, if $x < 0$.

then $f \searrow$ in $(-\infty,-2)$ and in $(-2,0)$.

(g). We then observe that:

$f \searrow$ in $(-2,0)$ then $f \nearrow (0,2)$.

Then, $f(0)$ is a local minimum

Since f has vertical asymptotes such that

$$f \xrightarrow[x\to 2^-]{} +\infty \qquad f \xrightarrow[x\to 2^+]{} -\infty,$$

there is no absolute extrema

(h). Concavity. We need $f''(x)$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f\right) = \frac{d}{dx}\left(\frac{4x}{(4-x^2)}\right) = 4\frac{d}{dx}\left(\frac{x}{(4-x^2)}\right)$$

$$= 4\left(\frac{(4-x^2)\cdot 1 - x(-2x)}{(4-x^2)^2}\right) = 4\left(\frac{4-x^2+2x^2}{(4-x^2)^2}\right)$$

i.e.

$$\boxed{\frac{d^2 f}{dx^2} = 4\left(\frac{4+x^2}{(4-x^2)^2}\right)}$$

Observe $(4+x^2) > 0$

$(4-x^2)^2 > 0$

(for $x \neq \pm 2$).

Then, $f(x)$ is convex i.e. concave up on each interval. $(-\infty,-2)$, $(-2,2)$, $(2,\infty)$

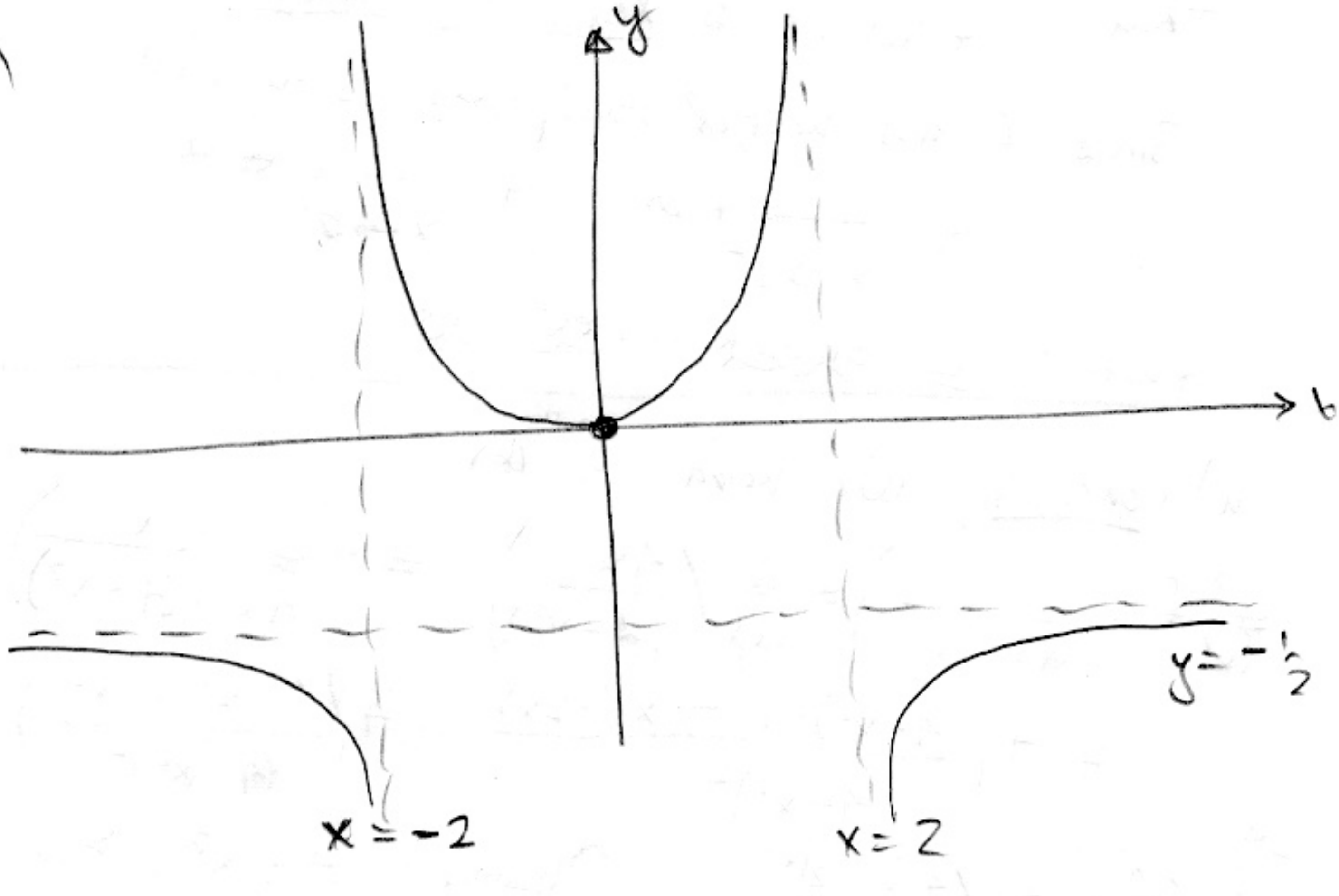
(i) Since f does not change its convexity, then, there is inflection points.

(j) We have to evaluate f'' at $x=0$.
We should get: $f''(0)>0$, for a minimum:

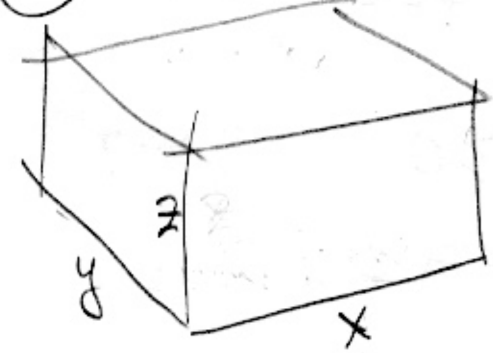
$$\frac{d^2}{dx^2}f(0) = 4\,\frac{(4+x^2)}{(4-x^2)^2}\bigg|_{x=0} = 4\,\frac{4}{4^2} > 0.$$

Then, $f(0)$ is a local minimum

(k)



(2) The container has a capacity $V_0 = 1000 m^3$.



The volume is computed as:

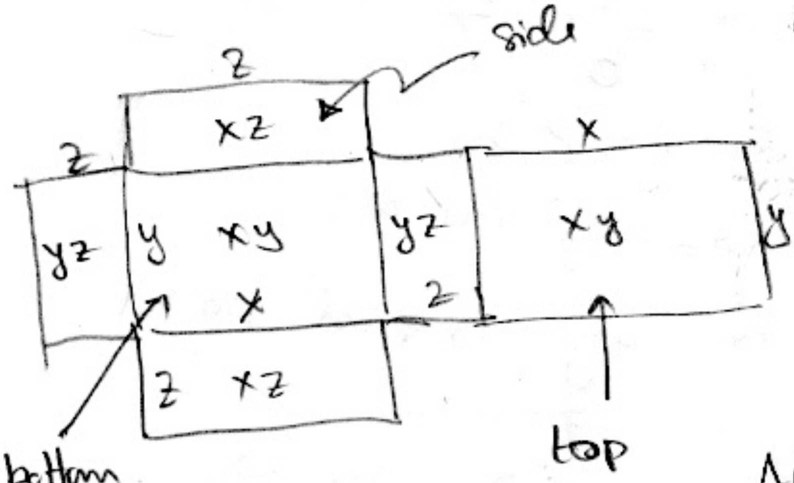
$$V = xyz.$$

Now length = 2(width)

$$y = 2x:$$

Hence: $V_0 = x(2x)z$ $\qquad 2x^2 z = V_0$ (fixed)

Now, the surface area is computed unfolding the box:



$$\text{Area} = 2xz + 2yz + 2xy$$
$$= 2(xz + yz + xy)$$

since: $y = 2x$.

$$A = 2(xz + (2x)z + x(2x))$$
$$A(x,y) = 2(3xz + 2x^2)$$

Then, the area is function of (x,y).

Now, using $2x^2 z = V_0 \Rightarrow z = \dfrac{V_0}{2x^2}$

$$A = 2\left(3x \frac{V_0}{2x^2} + 2x^2\right) = 2\left(\frac{3V_0}{2x} + 2x^2\right)$$

Then, we have to minimize $A(x) = 2\left(\dfrac{3V_0}{2x} + 2x^2\right)$

The cost $C(x) = 5A(x) = 10\left(\frac{3V_0}{2x} + 2x^2\right)$

<u>Now</u>

$\text{Dom}(C) = (0,\infty)$, since $x \geq 0$ is a length but $x \neq 0$

Now $C'(x) = 10\left(-\frac{3V_0}{2x^2} + 4x\right)$

(a) We need $C'(x) = 0$: $-\frac{3V_0}{2x^2} + 4x = 0$

$4x = \frac{3V_0}{2x^2} \Rightarrow x^3 = \frac{3V_0}{8} \Rightarrow x = \frac{\sqrt[3]{3V_0}}{2}$

(b) $C'(x)$ always exists on $(0,\infty)$

(c) There is no boundaries on $(0,\infty)$

Hence, $x_0 = \frac{\sqrt[3]{3V_0}}{2}$ is the only critical point.

Now, if $x < \frac{\sqrt[3]{3V_0}}{2} = x_0$, $x^3 < \frac{3}{8}V_0 \Rightarrow$

$\Rightarrow 4x < \frac{3V_0}{2x^2} \Rightarrow -\frac{3V_0}{2x^2} + 4x < 0$

$\Rightarrow C'(x) < 0$, then $C(x) \searrow$ in $(0, x_0)$

Similarly, if $x > \frac{\sqrt[3]{3V_0}}{2} = x_0$, $x^3 > \frac{3}{8}V_0$

$\Rightarrow 4x > \frac{3V_0}{2x^2} \Rightarrow 4x - \frac{3V_0}{2x^2} > 0 \Rightarrow C'(x) > 0$ in (x_0, ∞)

Then: $C(x) \searrow$ in $(0, x_0)$ and $C(x) \nearrow$ in (x_0, ∞)

Then $C(x)$ has a global minimum in $x = x_0$

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Similarly $$C''(x) = 10\left(-\frac{3V_0}{2}\left(-\frac{2}{x^3}\right) + 4\right)$$

$$= 10\left(\frac{3V_0}{x^3} + 4\right) > 0$$

$C(x)$ is positive, since $x > 0$.

Then $C''(x_0) > 0 \Rightarrow C(x_0)$ local minimum

Since $x = x_0$ is the only local minimum, it should be a global minimum.

The dimensions are found as follows

$$x_0 = \frac{\sqrt[3]{3V_0}}{2} = \frac{\sqrt[3]{3(1000)m^3}}{2} = \sqrt[3]{3}\,\frac{10\ m^3}{2}$$

$$= 5\sqrt{3}\ m^3 \approx 7.21\ m$$

$$y_0 = 2x_0 = 14.41\ m.$$

$$z_0 = \frac{V_0}{x_0 y_0} = \frac{1000\ m^3}{(5\sqrt{3})\,2(5\sqrt{3})\ m^2} = \frac{1000}{2 \cdot 5^2 \cdot 3}\ m$$

$$= \frac{1000}{150}\ m = \frac{100}{15}\ m = \frac{20}{3}\ m = 6.33\ m$$

$$C(x) = 5A(x) = 5 \cdot 2\left(x_0 y_0 + x_0 z_0 + y_0 z_0\right) \$.$$

$$= 10\left((5\sqrt{3})2(5\sqrt{3}) + (5\sqrt{3})\frac{20}{3} + 2(5\sqrt{3}) \cdot \frac{20}{3}\right)$$

$$= 10\left(150 + \frac{100}{\sqrt{3}} + \frac{200}{\sqrt{3}}\right) = 10\left(150 + \frac{300}{\sqrt{3}}\right)$$

$$= 10 \cdot 150\left(1 + \frac{2}{\sqrt{3}}\right) = 1500\left(\frac{\sqrt{3}+2}{\sqrt{3}}\right)$$

$$\approx 1500\left(1 + \frac{2}{\sqrt{3}}\right) \approx \underline{3{,}232.05 \$.}$$

(1) (a) I will assume I study Physical Engineering

(b) In Engineering, and in particular, in Physics, many systems look for the ~~minimum of energy~~

maximum of energy.

(c) The energy is the quantity to be maximized. The energy of a boiler (as an example), depends on the insulation parts. So, we will look fo insulation materials, and this would be our independent variable, the insulation coefficients of several materials

(d) As usual, we need to find

$$\mathcal{E}(i) = \text{Energy function of the isolation}$$

Find $\mathcal{E}'(i)$, the critical points, and check where $\mathcal{E}(i) \nearrow$ then $\mathcal{E}(i) \searrow$ to get a maximum energy