UAM-AZCAPOTZALCO $\boxed{\text{CÁLCULO INTEGRAL}}$ Martes, Septiembre 28, 2021

Examen #2 ANSWER KEY

① We need to compute:

$$\int \cot^5(x)\sin^4(x)dx = \int \frac{\cos^5(x)}{\sin^5(x)}\sin^4(x)\,dx =$$

$$= \int \frac{\cos^5(x)}{\sin x}\,dx = \int \frac{\cos^4(x)}{\sin x}\cdot\cos(x)\,dx$$

$$= \int \frac{(\cos^2 x)^2}{\sin x}\cos x\,dx = \int \frac{(1-\sin^2 x)^2}{\sin x}\cos x\,dx$$

Take $y(x)=\sin x$. then $\frac{dy}{dx}=\cos x$. then

$$= \int \frac{(1-y^2(x))}{y(x)}\frac{dy}{dx}(x)\,dx = \int \frac{(1-y^2)^2}{y}\,dy$$

By the Theorem of Change of Variable

$$= \int \frac{1-2y^2+y^4}{y}\,dy$$

$$= \int \left(\frac{1}{y}-2y+y^3\right)dy = \log|y| - y^2 + \frac{1}{4}y^4 + C$$

Returning to the original variable:

$$= \log|\sin x| - \sin^2(x) + \frac{1}{4}\sin^4(x) + C$$

② $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$ | Use: $x = \sec y$

Now $x^2 - 1 = \sec^2 y - 1$

$= \tan^2 y$.

Also: $\frac{dy}{dy} = \sec y \tan y$

Hence:
$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{1}{x^3(y)\sqrt{x^2(y)-1}}\,\frac{dx}{dy}(y)\,dy$$

by the Theorem of Change of variable:

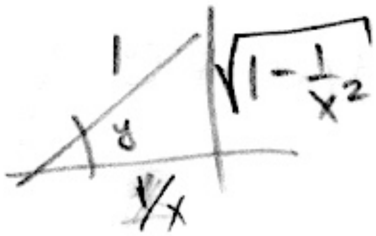
$$= \int \frac{1}{\sec^3 y\sqrt{\sec^2 y - 1}}\,\sec y \tan y\,dy$$

$$= \int \frac{1}{\sec^3 y\,\cancel{\sqrt{\tan^2 y}}}\,\sec y\,\cancel{\tan y}\,dy$$

$$= \int \frac{1}{\sec^2 y}\,dy = \int \cos^2 y\,dy = \int \frac{(1+\cos(2y))}{2}\,dy$$

Now
$$= \frac{1}{2}\int dy + \frac{1}{2}\int \cos 2y\,dy = \frac{y}{2} + \frac{1}{4}\sin 2y + C$$

$$= y/2 + \frac{2\sin y\cos y}{2} + C = \boxed{y/2 + \frac{\sin y\cos y}{2} + C}$$



$\cos y = \frac{1/x}{1} = \frac{1}{x}$; $\sin y = \sqrt{1-\cos^2 y}$

$y = \mathrm{Arccos}\left(\frac{1}{x}\right)$; $= \sqrt{1 - \frac{1}{x^2}}$

Hence.

$$\boxed{\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \mathrm{Arccos}\left(\frac{1}{x}\right) + \frac{1}{2x}\sqrt{1-\frac{1}{x^2}} + C}$$

③ $\dfrac{x^4+x^2}{(x^2-x)(x^4+4x^2+4)}$. Notice denominator has degree $2+4=6$, and numerator has degree 4. No need to divide polynomials.

Now

$$(x^2-x)(x^4+4x^2+4) = x(x-1)(x^2+2)^2$$

Then

$$\frac{x^4+x^2}{(x^2-x)(x^4+4x^2+4)} = \frac{x^4+x^2}{x(x-1)(x^2+2)^2} = \frac{x^3+x}{(x-1)(x^2+2)^2}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

where A, B, C, D, E are constants to be determined.

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(4) $\displaystyle\int \frac{5x-1}{(2x-1)(x+1)}\,dx$. deg (Num) = 1 < 2 = deg (Denom)
No need of polynomial division.

$$\frac{5x-1}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} = \frac{A(x+1) + B(2x-1)}{(2x-1)(x+1)}$$

Then

$$5x-1 = A(x+1) + B(2x-1)$$

Substitute at $\underline{x=-1}$

$$-5-1 = 0 + B(-2-1)$$

$$-6 = -3B \Rightarrow \boxed{B = 2}$$

$\underline{x = \frac{1}{2}}$

$$\frac{5}{2} - 1 = A\left(\frac{1}{2}+1\right) + 0$$

$$\frac{5-2}{2} = \frac{3}{2}A \Rightarrow 3A = 3$$

$$\boxed{A = 1}$$

thus.

$$\int \frac{5x-1}{(2x-1)(x+1)}\,dx = \int \left(\frac{1}{2x-1} + \frac{2}{x+1}\right)dx$$

$$= \frac{1}{2}\log|2x-1| + 2\log|x+1| + C$$

$$\boxed{\int \frac{5x-1}{(2x-1)(x+1)}\,dx = \log\left(\frac{\sqrt{|2x-1|}}{(x+1)^2}\right) + C}$$

(5) $\int_3^\infty \frac{1}{(x+3)^{3/2}}\,dx$. This is an improper integral

$$= \lim_{M\to\infty} \int_3^M (x+3)^{-3/2}\,dx = \lim_{M\to\infty} \left(-\frac{2}{1}\right)(x+3)^{-1/2}\Big|_3^M$$

$$= -2 \lim_{M\to\infty} \frac{1}{\sqrt{x+3}}\Big|_3^M = -2 \lim_{M\to\infty}\left(\frac{1}{\sqrt{M+3}} - \frac{1}{\sqrt{6}}\right)$$

$$= -2\left(0 - \frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}}$$