CÁLCULO DIFERENCIAL (UEA: 1112028)
EXAMEN DE RECUPERACIÓN (FORMA REMOTA).

UAM - AZCAPOTZALCO. (TRIMESTRE: INVIERNO DE 2020)
FECHA: JUEVES 28 DE OCTUBRE DE 2021. HORA: **16:00**.
HORA **LIMITE** DE ENTREGA: **19:30**

Nombre: ANSWER KEY.

- El examen consta de **OCHO** problemas con diferentes puntajes, para un total de 100 puntos. Tienen tres horas para resolverlos.
- El examen es **INDIVIDUAL** y se resuelve de forma **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo. Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- Para recibir puntaje: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) (**10 puntos.**) Diga qué ingeniería estudia. En sus propias palabras, describa un problema de su ingeniería que se pueda resolver usando las ideas de Cálculo Diferencial.

(2) (**10 puntos.**) Encuentre el dominio y calcule la derivada de:

(a) $q(x) = \left(\dfrac{2\sqrt{x}-1}{5\sqrt{x}-10}\right)^3$

(b) $r(x) = x\tan\left(\dfrac{\sin x + \cos x}{2}\right)$.

(3) (**15 puntos.**) Una partícula se desplaza en el primer cuadrante del plano cartesiano y a lo largo de la parábola $y = x^2$. Su coordenada x aumenta a una razón de 10 m/seg. ¿Qué tan rápido cambia la distancia de la partícula al origen cuando $x = 3$ m?

(4) (**10 puntos.**) Considere la función $f(x) = \dfrac{10}{x} - \dfrac{10}{8-x}$.

(a) El punto $(5, 4/3)$, ¿pertenece a la gráfica de f?

(b) Obtenga las ecuaciones de las rectas tangente y normal a la gráfica de f en el punto $(3, 4/3)$.

(5) **(15 puntos.)** Considere la función $g(x) = \dfrac{x^2+1}{(x-1)^2}$. Encontrar:

(a) dominio, raíces, paridad, periodicidad,
(b) asíntotas horizontales y verticales,
(c) puntos críticos,
(d) intervalos de crecimiento y decrecimiento,
(e) clasificación de los puntos críticos,
(f) concavidad y puntos de inflexión,
(g) máximos y mínimos absolutos y rango
(h) y además haga un esbozo de la gráfica.

(6) **(15 puntos.)** Calcule el volumen máximo del cilindro circular recto que se puede inscribir en un cono de 12 cm de altura y 4 cm de radio de base, de manera tal que los ejes del cilindro y el cono coincidan.

(7) **(10 puntos.)** Calcule las derivadas de las siguientes funciones.

(a) $s(x) = (\arctan x)^{\ln x^2}$
(b) $t(x) = e^{\arcsin x + \arccos x} + 3^5$.

(8) **(15 puntos.)** Para la función $A(x) = \arctan x$:

(a) Obtenga supolinomio de Taylor de grado 3 alrededor de $x = 1$.
(b) Aproxime $\arctan(1.1)$. Compare con su calculadora y escriba el resultado que ésta le dió.

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Examen de Recuperación. ANSWER KEY

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(1) En Ingeniería Industrial, por ejemplo, en un proceso de ensamblado y soldado, se requiere minimizar el tiempo para cumplir esta tarea.

Se requiere encontrar el tiempo como función de cada uno de los tiempos de ensamblado, y calcular el mínimo de esta función.

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(2) (a) $q(x) = \left(\dfrac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right)^3$

$\sqrt{x}$ requires $x \geq 0$

$5\sqrt{x} - 10 = 0$ implies $\sqrt{x} = \frac{10}{5} = 2 \Rightarrow x = 4$

We need $x \neq 4$.

$$\boxed{\text{Dom}(q) = [0, 4) \cup (4, \infty)}$$

$$\frac{dq}{dx} = \frac{d}{dx}\left(\frac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right)^3 = 3\left(\frac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right)^2 \frac{d}{dx}\left(\frac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right) \quad \text{by Chain rule}$$

$$= 3\left(\frac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right)^2 \frac{(5\sqrt{x} - 10)\frac{d}{dx}(2\sqrt{x} - 1) - (2\sqrt{x} - 1)\frac{d}{dx}(5\sqrt{x} - 10)}{(5\sqrt{x} - 10)^2} \quad \text{Quotient rule}$$

$$= 3\left(\frac{2\sqrt{x} - 1}{5\sqrt{x} - 10}\right)^2 \frac{(5\sqrt{x} - 10)\frac{2}{2\sqrt{x}} - (2\sqrt{x} - 1)\frac{5}{2\sqrt{x}}}{(5\sqrt{x} - 10)^2}$$

$$= \frac{3(2\sqrt{x} - 1)^2}{(5\sqrt{x} - 10)^2} \cdot \frac{10\sqrt{x} - 20 - 10\sqrt{x} + 5}{2\sqrt{x}(5\sqrt{x} - 10)^2}$$

$= 1 =$

i.e. $\boxed{\dfrac{dq}{dx} = -\dfrac{45}{2}\, \dfrac{(2\sqrt{x}-1)^2}{\sqrt{x}\,(5\sqrt{x}-10)^4}}$

(b) $r(x) = x \tan\left(\dfrac{\sin x + \cos x}{2}\right)$

For $\tan\theta$, we require $|\theta| < \dfrac{\pi}{2}$.

Now, $\left|\dfrac{\sin x + \cos x}{2}\right| < \dfrac{|\sin x| + |\cos x|}{2} < \dfrac{1+1}{2} = 1 < \dfrac{\pi}{2}$

and since sin, cos are valid $\forall x \in \mathbb{R}$, then $r(x)$ has no problems: $\boxed{\mathrm{Dom}(r) = \mathbb{R}}$

Now

$$\frac{dr}{dx} = \frac{d}{dx}\left(x \tan\left(\frac{\sin x + \cos x}{2}\right)\right) \overset{\text{Product rule}}{=} \left(\frac{dx}{dx}\right)\tan\left(\frac{\sin x + \cos x}{2}\right) + x\frac{d}{dx}\left(\tan\left(\frac{\sin x + \cos x}{2}\right)\right)$$

$$= \tan\left(\frac{\sin x + \cos x}{2}\right) + x \sec^2\left(\frac{\sin x + \cos x}{2}\right)\frac{d}{dx}\left(\frac{\sin x + \cos x}{2}\right), \text{ Chain rule}$$

$$\boxed{\frac{dr}{dx} = \tan\left(\frac{\sin x + \cos x}{2}\right) + x \sec^2\left(\frac{\sin x + \cos x}{2}\right)\left(\frac{\cos x - \sin x}{2}\right)}$$

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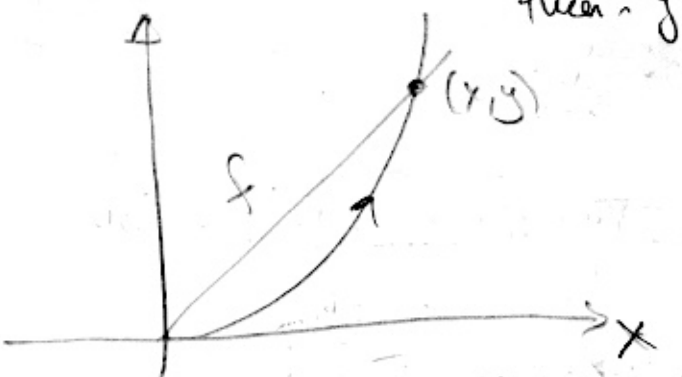
③ The particle has a position (x,y): Its distance to the origin is: $f(x) = \sqrt{x^2+y^2}$. Since $(x,y) \in$ parabola then: $y = x^2$. Hence:



$$f(x) = \sqrt{x^2+y^2} = \sqrt{x^2+(x^2)^2}$$
$$= \sqrt{x^2+x^4}$$

How fast does the particle go away from the origin?
We need to compute the derivative, with respect to time.

$$\frac{df}{dt} = \frac{d}{dx}(x^2+x^4)^{\frac{1}{2}} = \frac{1}{2}(x^2+x^4)^{-\frac{1}{2}}\frac{d}{dt}(x^2+x^4), \text{ Chain rule.}$$

$$= \frac{1}{2\sqrt{x^2+x^4}}(2x+4x^3)\frac{dx}{dt} = \boxed{\frac{x+2x^3}{\sqrt{x^2+x^4}}\left(\frac{dx}{dt}\right)}$$

Since: $\frac{dx}{dt} = \frac{10\,m}{sec}$:

$$\boxed{\frac{df}{dt} = \frac{10(x+2x^3)}{\sqrt{x^2+x^4}}\ m/sec.}$$

Units are important.

If $x = 3$:

$$\frac{df}{dt} = \frac{10(3+2\cdot 27)}{\sqrt{9+81}}\frac{m}{sec} = \frac{570}{\sqrt{90}}\frac{m}{sec} = \frac{570}{3\sqrt{10}}\frac{m}{sec}$$

$$\boxed{\frac{df}{dt} = \frac{190}{\sqrt{10}}\frac{m}{sec}} \approx 60.08\ m/sec.$$

4. $f(x) = \frac{10}{x} - \frac{10}{8-x}$

(a) Take $x=5$: $f(5) = \frac{10}{5} - \frac{10}{8-5} = 2 - \frac{10}{3} = \frac{6-10}{3} = -\frac{4}{3}$ $\neq +\frac{4}{3}$

Then, $\left(5, \frac{4}{3}\right)$ does not belongs to the graph.

(b) $f(3) = \frac{10}{3} - \frac{10}{8-3} = \frac{10}{3} - \frac{10}{5} = \frac{10}{3} - 2 = \frac{10-6}{3} = +\frac{4}{3}$ ✓

$\left(3, \frac{4}{3}\right)$ is on the graph

Now: $f'(x) = -\frac{10}{x^2} - \frac{10}{(8-x)^2}\frac{d}{dx}(8-x) = -\frac{10}{x^2} - \frac{10}{(8-x)^2}(-1)$

$= -\frac{10}{x^2} + \frac{10}{(8-x)^2}$.

$f'(3) = -\frac{10}{3^2} + \frac{10}{(8-3)^2} = -\frac{10}{9} + \frac{10}{25} = \frac{-250+90}{225} = \frac{-160}{225}$

$= -\frac{32}{45}$ this is the slope of the tangent line.

whose equation at $\left(3, \frac{4}{3}\right)$ is $\boxed{y - \frac{4}{3} = -\frac{32}{45}(x-3)}$

or $y-4 = -\frac{32}{15}(x-3)$ or $y = -\frac{32}{15}x + \frac{32}{15} + 4 \Rightarrow \boxed{y = -\frac{32}{15}x + \frac{52}{5}}$

The normal line has slope $m = -\frac{1}{f'(3)} = \frac{45}{32}$. Hence, the normal line has equation

$\boxed{y - \frac{4}{3} = \frac{45}{32}(x-3)}$ i.e. $y = \frac{45}{32}x - \frac{135}{32} + \frac{4}{3}$ $\boxed{y = \frac{45}{32}x - \frac{67}{24}}$

Since: $-\frac{135}{32} + \frac{4}{3} = \frac{-396+128}{96} = -\frac{268}{96} = \frac{-67}{24}$

(5) $g(x) = \dfrac{x^2+1}{(x-1)^2}$

(a) Domain: Require $(x-1) \neq 0 \Rightarrow x \neq -1$ $\boxed{Dom(g) = \mathbb{R} \setminus \{1\}}$

Since $x^2+1 > 0$, $g(x) > 0$, there is no roots.

$g(-x) = \dfrac{(-x)^2+1}{(-x-1)^2} = \dfrac{x^2+1}{(x+1)^2} \neq g(x)$ Neither even nor odd.

It is not periodic

(b) $\displaystyle\lim_{x \to \pm\infty} \frac{x^2+1}{(x-1)^2} = \lim_{x \to \pm\infty} \frac{x^2(1+1/x^2)}{x^2(1-1/x)^2} = \lim_{x \to \pm\infty} \frac{1+1/x^2}{(1+1/x)^2}$

$= \dfrac{1}{1} = 1$ Then, $\boxed{y = 1 \text{ horizontal asymptote}}$

Now:

$\displaystyle\lim_{x \to 1^{\pm}} \frac{x^2+1}{(x-1)^2} = \lim_{x \to 1^{\pm}} \frac{2}{(x-1)^2} = \infty$

$\boxed{\text{Then } x = 1 \text{ is a vertical asymptote}}$

(c) We need $\dfrac{dg}{dx} = \dfrac{d}{dx}\left(\dfrac{(x^2+1)}{(x-1)^2}\right) = \dfrac{(x-1)^2(x^2+1)' - (x^2+1)((x-1)^2)'}{(x-1)^4}$

$= \dfrac{2x(x-1)^2 - (x^2+1)(2)(x-1)}{(x-1)^4}$

$= \dfrac{2[x(x-1) - (x^2+1)](x-1)}{(x-1)^4}$

$= \dfrac{2[x^2 - x - x^2 - 1]}{(x-1)^3}$

$\boxed{\dfrac{dg}{dx} = \dfrac{-2(x+1)}{(x-1)^3}}$

Critical points: (i) Boundary points: there is no boundaries

(ii) $\frac{dg}{dx}$ does not exist.

$g'(x)$ does not exist on $x = 1$, but $1 \notin Dom(g)$

Hence, no problem with $x = 1$

(iii) $\frac{dg}{dx} = 0$; here $\frac{-2(x+1)}{(x-1)^3} = 0$ only if $\underline{x = -1}$

Hence, $x = -1$ is the only critical point.

(d) Monotonicity. Divide the real interval into

$(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$, This are intervals of continuity of $\frac{dg}{dx}$.

since we have problems at : $x = 1$, and $x = -1$ is a critical point.

Take: $x = -2$, $x = 0$, $x = 2$; evaluate $\frac{dg}{dx} = \frac{-2(x+1)}{(x-1)^3}$

$\frac{dg}{dx}(-2) = \frac{-2(-1)}{(-3)^3} = \frac{-2}{27} < 0$; $g \searrow$ on $(-\infty, -1)$.

$\frac{dg}{dx}(0) = \frac{-2}{(-1)^3} = 2 > 0$: $g \nearrow$ on $(-1, 1)$

$\frac{dg}{dx}(+2) = \frac{-2(3)}{1^3} = -6 < 0$: $g \searrow$ on $(1, \infty)$.

(e) Then g has a local minimum on $x = -1$:

with minimum value $g(-1) = \frac{(-1)^2+1}{(-1-1)^2} = \frac{2}{4} = \frac{1}{2}$.

(f) Second derivative:

$$\frac{d^2}{dx^2} g = -2 \frac{d}{dx}\left(\frac{x+1}{(x-1)^3}\right) = -2 \frac{(x-1)^3 (x+1)' - (x+1)((x-1)^3)'}{(x-1)^6}$$

$$= -2 \frac{(x-1)^3 - 3(x+1)(x-1)^2}{(x-1)^6} = -2\left(\frac{(x-1) - 3(x+1)}{(x-1)^6}\right)(x-1)^2$$

$$= -2\left(\frac{x-1-3x-3}{(x-1)^4}\right) = -2\left(\frac{-2x-4}{(x-1)^4}\right)$$

$$= 4 \frac{x+2}{(x-1)^4}$$

If $x > -2$ then, $\frac{d^2}{dx^2} g > 0$, then it is concave up. (except at $x = 1$)

If $x < -2$, then $\frac{d^2}{dx^2} g < 0$, then g is concave down

Since there is a change of concavity at $x = -2$, then $x = -2$ is an inflexion point.

(g) Since $g(x) \xrightarrow[x \to 1]{} \infty$, there is no global max

Since $g(-1) = \frac{1}{2}$ is a local min, and:

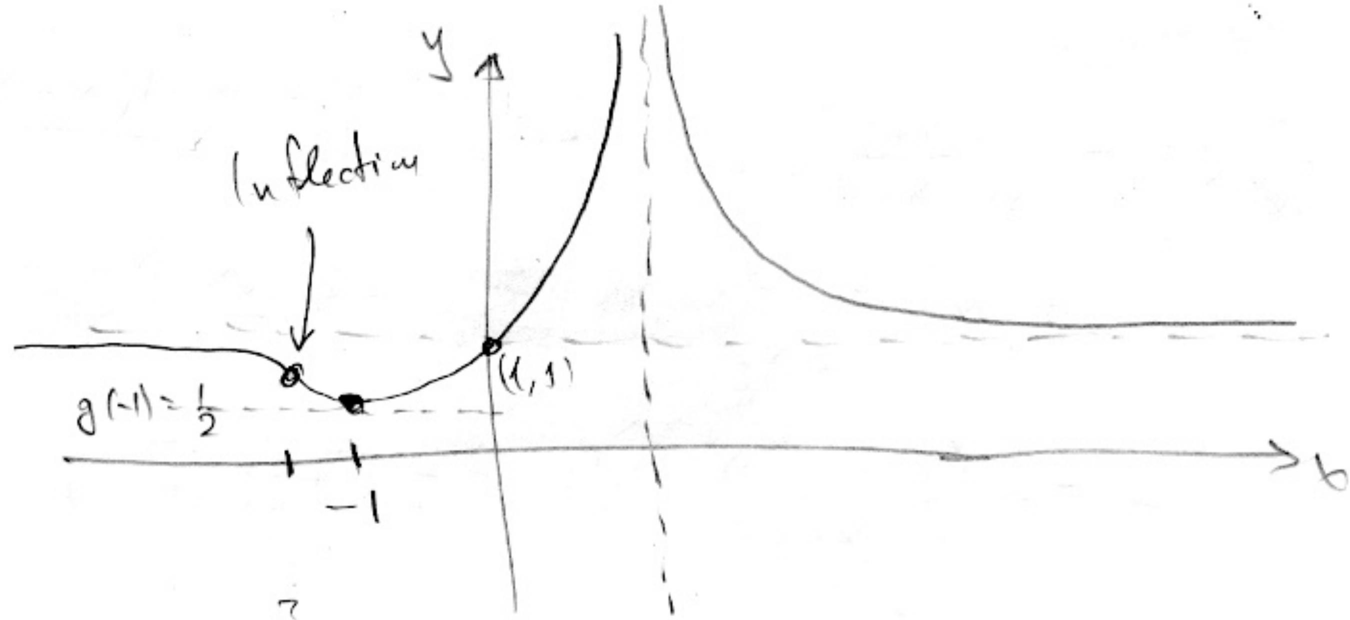
$g \searrow$ on $(-\infty, -1)$, $g \nearrow$ on $(-1, 1)$

Then, $g(-1) = \frac{1}{2}$ is minimum on $(-\infty, 1)$.

Now, for $x > 1$, $g \longrightarrow 1$ as $x \longrightarrow \infty$, and $g \searrow$ on $(1, \infty)$, then $g(x) > \frac{1}{2}$ on $(1, \infty)$

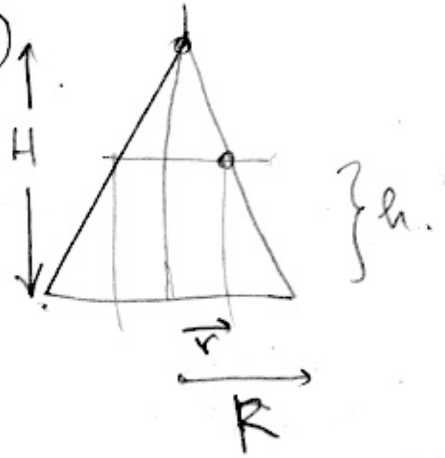
Hence $\boxed{g(-1) = \frac{1}{2} \text{ is a global min}}$

(h)



$$g(0) = \frac{0^2+1}{(0-1)^2} = 1$$

(6)

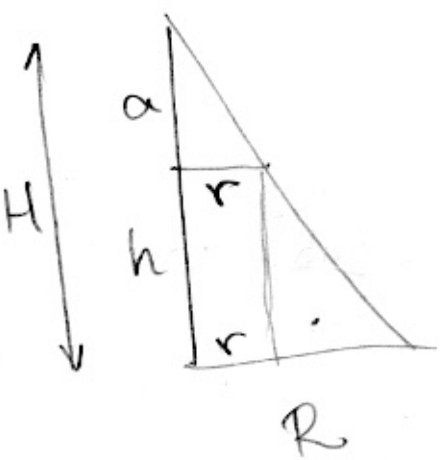


Volume of cylinder of radius r and heigth h:

$$\boxed{V(r,h) = \pi r^2 h}$$

We need a relation in between r, h

We do this with similar triangles.



$$\frac{r}{a} = \frac{R}{H} \Rightarrow r = \frac{R}{H} a$$

But $a + h = H \Rightarrow a = H - h$.

$$\Rightarrow \quad r = \frac{R}{H}(H-h)$$

or $\quad \frac{H}{R} r = H - h \Rightarrow \boxed{h = H\left(1 - \frac{r}{R}\right)}$

Hence: $\boxed{V(r) = \pi r^2 H\left(1 - \frac{r}{R}\right)}$ $\qquad \boxed{\frac{d^2V}{dr^2} = 2\pi H\left(1 - \frac{3r}{R}\right)}$

or. $\boxed{V(r) = \pi H\left(r^2 - \frac{r^3}{R}\right)}$

$\mathrm{Dom}(V) = [0, R]$, since $0 \leq r \leq R$.

Now $\frac{dV}{dr} = \pi H\left(2r - \frac{3r^2}{R}\right) = 2\pi H r\left(1 - \frac{3r}{2R}\right)$.

Critical points (i) Boundary points $r = 0$, $r = R$

(ii) $\frac{dV}{dr}$ does not exist, but it is defined on $[0, R]$: Always exists

(iii) $\frac{dV}{dr} = 0$ if $2\pi H r\left(1 - \frac{3r}{2R}\right) = 0$.

Then $r_1 = 0$, or, $1 - \frac{3r_2}{2R} = 0 \Rightarrow r_2 = \frac{2R}{3}$

We have three critical points: on $[0,R]$

$r_1 = 0$, $r_2 = \frac{2R}{3}$, $r_3 = R$

Then $V(0) = \pi 0^2 H\left(1 - \frac{0}{R}\right) = 0$

$V\left(\frac{2R}{3}\right) = \pi\left(\frac{2R}{3}\right)^2 H\left(1 - \frac{1}{R}\frac{2R}{3}\right) = \frac{4\pi R^2 H}{9}\left(1 - \frac{2}{3}\right)$

$= \frac{4\pi R^2 H}{27}$.

$V(R) = \pi R^2 H\left(1 - \frac{R}{R}\right) = \pi R^2 H \cdot 0 = 0$.

Since V is continuous on $[0,R]$ it reaches its extrema where r is a critical point. Then.

$\boxed{\max(V) = \frac{4\pi R^2 H}{27}.}$ Here $H = 12$ cm
$R = 4$ cm:

$\boxed{\min(V) = 0 \text{ cm}^3}$ $R^2 H = 16 \cdot 12 = 192$:

Hence $\max(V) = \frac{4\pi \cdot 192}{27} = \frac{4\pi \cdot 64}{9} \Rightarrow \boxed{\max(V) = \frac{256\pi}{9} \text{ cm}^3}$

Now, Local max because 2nd derivative test:

$\frac{d^2V}{dr^2}\left(\frac{2R}{3}\right) = 2\pi H\left(1 - \frac{3}{R}\left(\frac{2R}{3}\right)\right) = 2\pi H(-1) < 0$

$\max(V) = 89.36 \text{ cm}^3$

7 (a) $\ln s(x) = 2\ln x \ln(\operatorname{Arctan} x)$

$$\frac{s'}{s} = \frac{2}{x}\ln(\operatorname{Arctan} x) + 2\ln x \, \frac{1}{\operatorname{Arctan} x} \cdot \frac{1}{1+x^2}$$

$$\Rightarrow s'(x) = (\operatorname{Arctan} x)^{\ln x^2}\left(\frac{2\ln(\operatorname{Arctan} x)}{x} + \frac{2\ln x}{(1+x^2)(\operatorname{Arctan} x)}\right)$$

(b) $\frac{d t(x)}{dx} = \frac{d}{dx}\left(e^{\arcsin x + \arccos x} + 3^5\right)$

$$= \frac{d}{dx}\left(e^{\arcsin x + \arccos x}\right) + 0$$

It is enough to compute the derivative of

$$T(x) = e^{\arcsin x + \arccos x}$$

$$\ln(T) = \ln e^{(\arcsin x + \arccos x)}$$

$$= \arcsin x + \arccos x$$

The derivative:

$$\frac{T'}{T} = \frac{d}{dx}(\arcsin x + \arccos x)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{(-1)}{\sqrt{1-x^2}} = 0$$

Then $T'(x) = 0$ i.e. $\boxed{\frac{d}{dx}\left(e^{\operatorname{Arcsin} x + \operatorname{Arccos} x} + 3^5\right) = 0}$

⑧ (a) $A(x) = \operatorname{Arctan} x$

$$\frac{dA}{dx} = \frac{1}{1+x^2}$$

$$\frac{d^2A}{dx^2} = \frac{-2x}{(1+x^2)^2}$$

$$\frac{d^3A}{dx^3} = \frac{(1+x^2)^2(-2) - (-2x)\left(\left((1+x^2)^2\right)'\right)}{(1+x^2)^4}$$

$$= \frac{-2(1+x^2)^2 + 2x\, 2(1+x^2)\, 2x}{(1+x^2)^4}$$

$$= \frac{-2(1+x^2)\left[(1+x^2) - 4(1+x^2)x^2\right]}{(1+x^2)^4}$$

$$= \frac{-2\left[1+x^2-4x^2-4x^4\right]}{(1+x^2)^3}$$

$$= \frac{-2(1+x^2-4x^2-4x^4)}{(1+x^2)^3}$$

Evaluate at $x=1$:

$A(1) = \operatorname{Arctan} 1 = \frac{\pi}{4}$, $A'(1) = \frac{1}{2}$, $A''(1) = \frac{-2}{(1+1)^2} = -\frac{1}{2}$

$A'''(1) = 12/8 = 3/2$

Hence: $P(x) = A(1) + A'(1)(x-1) + \frac{A''(1)}{2}(x-1)^2 + \frac{A'''(1)}{3!}(x-1)^3$

$$\boxed{P(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{4}(x-1)^3}$$

This is the polynomial of degree 3! Complete to order 2.

(b) $P(1.1) = \frac{\pi}{4} + \frac{1}{2}\cdot\frac{1}{10} - \frac{1}{4}\frac{1}{100} = \frac{\pi}{4} + \frac{1}{20} - \frac{1}{400} = \boxed{\frac{\pi}{4} + \frac{19}{400}}$

$\boxed{\operatorname{Arctan}(1.1) \approx 0.832898}$ Using calculator: 0.83298126 Since! 666.