UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
CÁLCULO INTEGRAL
TRIMESTRE: OTOÑO DE 2021. PEER.

EXAMEN # 2.
FECHA: **VIERNES 14 DE ENERO DE 2022.**
DE 16:00 A 17:30 HORAS. ENTREGA: DE 17:30 A 18:00 HORAS

Nombre: SOLUTION KEY.

Instrucciones:

- El examen consta de **SIETE** problemas con diferentes puntajes.
- Tiene una (1) hora y treinta (30) minutos para resolver este examen.
- El examen es **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Pueden usar sus libros, apuntes y una calculadora sencilla o graficador sencillo. Cite cuando use libro, apuntes o su calculadora. Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- **Para recibir puntaje**: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(10 puntos)** ¿Es la integral $\displaystyle\int_{-10}^{+10} \frac{1}{z^4(36-z^2)}\,dz$ propia o impropia? Dé todas las razones.

(2) **(15 puntos)** Calcule la integral, de -1 a 0 (cero), de $8\dfrac{e^{2/z}}{z^3}$ respecto a la variable z.

(3) **(15 puntos)** Encuentre **únicamente la forma** de las fracciones parciales de la siguiente función racional.

$$\frac{2z^3+3z^2}{(z^4-16)(z^4+4z^2+4)}$$

(4) **(25 puntos)** Calcule la antiderivada de $\dfrac{1}{\cot^3(4z)}$ respecto a la variable z.

(5) **(25 puntos)** Calcule la antiderivada de $8\dfrac{z^{-2}}{\sqrt{(z^2/4)-4}}$ respecto a la variable z.

(6) **(10 puntos)** Calcule la antiderivada de $\dfrac{z^2}{z^4-16}$ respecto a la variable z.

UAM-AZCAPOTZALCO CÁLCULO INTEGRAL Viernes, Enero 14, 2022

Examen #2. ANSWER KEY

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(1). The integral is improper since

$$f(z) = \frac{1}{z^4(36-z^2)} = \frac{1}{z^4(6-z)(6+z)}$$

and $\lim_{\to 0} f(\,) = \infty$, $\lim_{\to 6} f(z) = \pm\infty$

and $\lim_{\to -6} f(\,) = \pm\infty$, i.e. f has vertical asymptotes at $x=-6$, $x=0$ and $x=+6$

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(2) We are required to compute the integral

$\displaystyle\int_{-1}^{0} 8\,\frac{e^{2/z}}{z^3}\,dz$. By the change $y = \frac{z}{2}$: $\Rightarrow \left\{ \begin{array}{l} z = 2y \\ \frac{dz}{dy} = 2. \end{array} \right.$

$$= \int_{-1/2}^{0} 8\,\frac{e^{1/y}}{(2y)^3}\,2\,dy = 2\int_{-1/2}^{0} \frac{e^{1/y}}{y^3}\,dy.$$

Now, let $y = \frac{1}{x}$ $\Rightarrow$ $\frac{dy}{dx} = -\frac{1}{x^2}$ and $\frac{dx}{dy} = -\frac{1}{y^2}$

$$= -2\int_{-1/2}^{0} \frac{1}{y}\,e^{1/y}\left(-\frac{1}{y^2}\right)dy = -2\int_{-1/2}^{0} \frac{1}{y}\,e^{1/y}\left(\frac{dx}{dy}\right)dy$$

$$= -2\int_{-2}^{-\infty} x\,e^x\,dx,$$ by substitution rule and $x = \frac{1}{(-1/2)} = 2$, $x \xrightarrow[y\to 0^-]{} -\infty$

$=1=$

$$= 2\int_{-\infty}^{-2} x\, e^{x}\, dx = 2\int_{-\infty}^{-2} x\,(e^{x})'\, dx$$

$$= 2\left[x e^{x} \Big|_{-\infty}^{-2} - \int_{-\infty}^{-2} (x)' e^{x}\, dx \right]$$ by integration by parts.

$$= 2\left[x e^{x} \Big|_{-\infty}^{-2} - \int_{-\infty}^{-2} e^{x}\, dx \right]$$

$$= 2\left[x e^{x} - e^{x} \right]_{-\infty}^{-2} = 2\,(x-1)e^{x} \Big|_{-\infty}^{-2}$$

$$= 2(-2-1)e^{-2} - \lim_{x \to -\infty} 2(x-1)\, e^{x}$$

Since $\dfrac{\text{Polynomial}(x)}{e^{x}} \xrightarrow[x \to +\infty]{} 0$, , the limit $= 0$

Hence $$\boxed{\int_{-1}^{0} 8\,\frac{e^{2/z}}{z^{3}}\, dz = -6e^{-2}}$$

(3) Observe we can write the rational function as follows:

$$\frac{2z^3+3z^2}{(z^4-16)(z^4+4z^2+4)} = \frac{2z^3+3z^2}{(z-2)(z+2)(z^2+4)(z^2+2)^2}$$

Since $\deg(P) = 3 < 8 = \deg(Q)$, there is no need of polynomial division. Then

$$= \underbrace{\frac{A}{z-2} + \frac{B}{z+2}}_{\text{They have power} = 1\text{, and they have no repeated roots}} + \underbrace{\frac{Cz+D}{z^2+4}}_{\text{It has power 2, it is non-reducible and has power 1}} + \underbrace{\frac{Ez+F}{z^2+2} + \frac{Gz+H}{(z^2+2)^2}}_{\text{Since it has power 2, and it is irreducible and has power 2.}}$$

(4) $$\int \frac{1}{\cot^3(4z)}\,dz = \frac{1}{4}\int \frac{1}{\cot^3(4z)}\,4\,dz,$$

Using $y = 4z$, $\frac{dy}{dz} = 4$, and by the Substitution rule

$$= \frac{1}{4}\int \frac{1}{\cot^3(y(z))}\frac{dy}{dz}\,dz = \frac{1}{4}\int \frac{1}{\cot^3(y)}\,dy$$

$$= \frac{1}{4}\int \tan^3 y\,dy = \frac{1}{4}\int \tan y\,(\tan^2 y)\,dy$$

$$= \frac{1}{4}\int \tan y\,(\sec^2 y - 1)\,dy$$

$$= \frac{1}{4}\int \tan y\,(\sec^2 y)\,dy - \frac{1}{4}\int \tan y\,dy.$$

Using: $u = \tan y$; $\frac{du}{dy} = \sec^2 y$, then

$$= \frac{1}{4}\int \tan y\,\frac{du}{dy}\,dy - \frac{1}{4}\int \tan y\,dy.$$

$$= \frac{1}{4}\int u\,\frac{du}{dy}\,dy - \frac{1}{4}\int \tan y\,dy$$

$$= \frac{1}{4}\int u\,du + \frac{1}{4}\int \frac{-\sin y}{\cos y}\,dy$$

$$= \frac{1}{8}u^2 + \frac{1}{4}\log|\cos y| + C$$

$$= \frac{1}{8}\tan^2 y + \frac{1}{4}\log|\cos y| + C$$

$$\boxed{\int \frac{1}{\cot^3(4z)}\,dz = \frac{1}{8}\tan^2(4z) + \frac{1}{4}\log|\cos(4z)| + C}$$

(5) $$\int \frac{8z^{-2}}{\sqrt{z^2/4-4}}\,dz = \int \frac{8\ z^{-2}}{\frac{1}{2}\sqrt{z^2-16}}\,dz = 16\int \frac{z^{-2}}{\sqrt{z^2-16}}\,dz.$$

Change of variables: $z = 4\sec\theta \qquad \frac{dz}{d\theta} = 4\sec\theta\tan\theta.$

$$= 16\int \frac{(z(\theta))^{-2}}{\sqrt{z^2(\theta)-16}}\,\frac{dz}{d\theta}(\theta)\,d\theta = 16\int \frac{(4\sec\theta)^{-2}}{\sqrt{16\sec^2\theta-16}}\,4\sec\theta\tan\theta\,d\theta$$

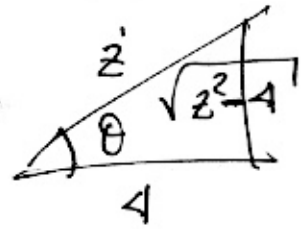
Substitution rule

$$= \frac{16\cdot 4^{-2}\cdot 4}{4}\int \frac{\sec^{-2}\theta\ \sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}} = 1\int \frac{\sec^{-2}\theta\ \sec\theta\tan\theta\,d\theta}{\tan\theta}$$

$\sec^2\theta = 1+\tan^2\theta$

$$= \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C.$$

Now $\sec\theta = \frac{z}{4} \Rightarrow \cos\theta = \frac{4}{z} \Longrightarrow$



$\Rightarrow \sin\theta = \frac{\sqrt{z^2-4}}{z}.$

Then.
$$\boxed{\int \frac{8z^{-2}}{\sqrt{z^2/4-4}}\,dz = \frac{\sqrt{z^2-4}}{z} + C}$$

⑥ This integral can be solve with two different methods

Method ① By partial fractions

$$\frac{z^2}{z^4-16} = \frac{z^2}{(z^2-4)(z^2+4)} = \frac{z^2}{(z-2)(z+2)(z^2+4)}$$

$$= \frac{A}{z-2} + \frac{B}{z+2} + \frac{Cz+D}{z^2+4}.$$

$$= \frac{A(z+2)(z^2+4) + B(z-2)(z^2+4) + (Cz+D)(z^2-4)}{(z-2)(z+2)(z^2+4)}$$

Then:

$$z^2 = A(z+2)(z^2+4) + B(z-2)(z^2+4) + (Cz+D)(z^2-4)$$

Take $z=2$

$$4 = A\cdot 4(8) + 0 + 0 \Rightarrow 1 = 8A \quad \boxed{A = \frac{1}{8}}$$

Take: $z=-2$

$$4 = A\cdot 0 + B(-4)8 + 0 \Rightarrow 1 = -B\cdot 8 \quad \boxed{B = \frac{-1}{8}}$$

Now take $z=0$

$$0 = A\cdot 2\cdot 4 + B(-2)(4) + D(-4).$$

$$8A - 8B - 4D = 0$$

$$1 - (-1) - 4D = 0 \Rightarrow 4D = 2 \Rightarrow \boxed{D = \frac{1}{2}}$$

Take $z=1$

$$1 = A\cdot 3\cdot 5 + B(-1)(5) + (C+D)(1-4)$$

$$15A - 5B + (-3)(C+D) = 1$$

$$\frac{15}{8} + \frac{5}{8} - 3C - 3D = 1$$

$$\frac{20}{8} - 3C - \frac{3}{2} = 1 \Rightarrow 3C = \frac{20}{8} - \frac{3}{2} - 1$$

$$\Rightarrow 3C = \frac{5}{2} - \frac{3}{2} - \frac{2}{2}$$

$$\Rightarrow \boxed{C = 0}$$

$$\Rightarrow 3C = 0$$

Hence:

$$\int \frac{z^2}{z^4 - 16}\, dz = \int \frac{1}{8}\frac{1}{z-2} - \frac{1}{8}\frac{1}{z+2} + (0)\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4}\, dz$$

$$= \frac{1}{8}\int \frac{1}{z-2}\, dz - \frac{1}{8}\int \frac{1}{z+2}\, dz + 0 + \frac{1}{2}\int \frac{1}{z^2+4}\, dz$$

$$= \frac{1}{8}\operatorname{Log}|z-2| - \frac{1}{8}\operatorname{Log}|z+2| + 0 + \frac{1}{2}\int \frac{1}{z^2+4}\, dz$$

$$= \frac{1}{8}\operatorname{Log}\left|\frac{z-2}{z+2}\right| + 0 + \frac{1}{2}\frac{1}{2}\operatorname{Arctan}\left(\frac{z}{2}\right) + C$$

$$\boxed{= \frac{1}{8}\operatorname{Log}\left|\frac{(z-2)}{(z+2)}\right| + \frac{1}{4}\operatorname{Arctan}\left(\frac{z}{2}\right) + C}$$

Method ②

$$\int \frac{z^2}{z^4-16}\,dz = \frac{1}{2}\int \frac{z^2+z^2}{(z^2-4)(z^2+4)}\,dz =$$

$$= \frac{1}{2}\int \frac{(z^2-4)+(z^2+4)\,dz}{(z^2-4)(z^2+4)} = \frac{1}{2}\left[\int \frac{1}{z^2+4}dz + \int \frac{1}{z^2-4}dz\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{z}{2}\right) + \frac{1}{4}\operatorname{Log}\left(\frac{z-2}{z+2}\right) + C\right]$$

$$\boxed{\int \frac{z^2}{z^4-16}\,dz = \frac{1}{4}\operatorname{Arctan}\left(\frac{z}{2}\right) + \frac{1}{8}\log\left(\frac{z-2}{z+2}\right) + C}$$

Since:

$$\int \frac{1}{z^2-4}\,dz = \int \frac{1}{4}\left(\frac{1}{z-2} - \frac{1}{z+2}\right)dz =$$

$$= \frac{1}{4}\left(\operatorname{Log}|z-2| - \operatorname{Log}|z+2|\right) = \frac{1}{4}\log\frac{|z-2|}{|z+2|} + C$$