UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
CÁLCULO DIFERENCIAL
TRIMESTRE: PRIMAVERA DE 2022.

EXAMEN # 1. ← A

FECHA: **MARTES 9 DE AGOSTO DE 2022.** DE 14:30 A 16:00 HORAS.

Nombre: ANSWER KEY

Instrucciones:

- El examen consta de **SEIS** problemas con diferentes puntajes, para un total de **100 puntos**.
- Tiene una (1) hora y treinta (30) minutos para resolver este examen.
- El examen es **INDIVIDUAL** y se resuelve de forma **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- Si salen fracciones o raíces, NO las convierta a decimales con su calculadora. Déjelas indicadas (a menos que vaya a estimar valores).
- **Para recibir puntaje**: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(10 puntos.)** Usando la definición de derivada, encuentre la ecuación de la recta tangente a la gráfica de

$$f(x) = \sqrt{x-1}, \quad \text{en el punto correspondiente a} \quad x = 5.$$

(2) **(20 puntos.)** Considere las funciones $f(x) = \cos x$, $g(x) = x$, y $h(x) = x^2 + 1$. Calcule la derivada de:

$$F(x) = \frac{f(x)}{g(x)} + \frac{h(x)}{f(x)}.$$

(3) **(20 puntos.)** Dadas $f(x) = \tan(5x)$ y $g(x) = 1/x$

(a) escriba

$$F(x) = \cos(5x)(f \circ g)(x) + 5^9.$$

(b) calcule la derivada de F.

(4) **(20 puntos.)** Encuentre la ecuación de la recta normal (*i.e.*, perpendicular) a la elipse

$$x^2/4 + xy + 4y^2 = 3,$$

en el punto (2,1/2).

(5) **(20 puntos.)** Usted se encuentra en la torre Latinoamericana a 160 metros sobre el nivel del suelo. Desde all arriba, usted suelta una pelota de esponja. ¿A qué velocidad caerá al piso? Suponga que la aceleracin de gravedad es de 10 metros/segundo^2.

(6) **(10 puntos.)** Calcule la linealización de la función $f(x) = (x-1)^{1/2}$, en $x = 5$. Use la definición de derivada.

AM- AZCAPOTZALCO | Examen #1-A | Martes, Agosto 9, 2022

CÁLCULO DIFERENCIAL ANSWER KEY

(1) The definition of derivative function is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Here, the function under consideration is: $f(x) = \sqrt{x-1}$

Then,

$$(\sqrt{x-1})' = \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)-1} - \sqrt{x-1}}{\Delta x}, \text{ by definition.}$$

Rationalizing, multiplying by 1.

$$= \lim_{\Delta x \to 0} \left(\frac{\sqrt{(x+\Delta x)-1} - \sqrt{x-1}}{\Delta x}\right) \cdot \left(\frac{\sqrt{(x+\Delta x)-1} + \sqrt{x-1}}{\sqrt{x+\Delta x-1} + \sqrt{x-1}}\right)$$

Difference of squares

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{(x+\Delta x)-1}\right)^2 - \left(\sqrt{x-1}\right)^2}{\Delta x\left(\sqrt{(x+\Delta x)-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - 1 - (x-1)}{\Delta x\left(\sqrt{(x+\Delta x)-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{(x-1) + \Delta x - (x-1)}{\Delta x\left(\sqrt{(x+\Delta x)-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-1} + \sqrt{x-1}}$$

and evaluating the limit:

$= 1 =$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

Then

$$\frac{d}{dx}\left(\sqrt{x-1}\right) = \frac{1}{2\sqrt{x-1}}$$

Now, the slope of the tangent line is the derivative evaluated at the x-value required, here, $x = 5$.

$$m = f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}.$$

Now, the graph of f passes through $(5, f(5))$.
Here $f(5) = \sqrt{5-1} = \sqrt{4} = 2$, i.e., $(5, 2) = (x_0, y_0)$

Then, the equation of a line is

$$y - y_0 = m(x - x_0)$$

$$\boxed{y - 2 = \frac{1}{4}(x - 5).}$$ slope-point form.

We can simplify:

$$y = \frac{1}{4}x - \frac{5}{4} + 2 = \frac{1}{4}x - \frac{5}{4} + \frac{8}{4}.$$

ie. $$\boxed{y = \frac{1}{4}x + \frac{3}{4}}$$ y-intercept form

We have to use the sum, quotient and power rules:

$$\frac{d}{dx}\left(\frac{\cos x}{x} + \frac{x^2+1}{\cos x}\right) = \frac{d}{dx}\left(\frac{\cos x}{x}\right) + \frac{d}{dx}\left(\frac{x^2+1}{\cos x}\right)$$

$$= \frac{x\,(\cos x)' - (\cos x)(x)'}{x^2} + \frac{\cos x\,(x^2+1)' - (x^2+1)(\cos x)'}{\cos^2 x}$$

$$= \frac{x(-\sin x) - (\cos x)\,1}{x^2} + \frac{(\cos x)(2x) - (x^2+1)(-\sin x)}{\cos^2 x}$$

$$= \frac{-x\sin x - \cos x}{x^2} + \frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}.$$

$$= -\frac{x\sin x + \cos x}{x^2} + \frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}$$

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③ Here $f(x) = \tan(5x)$, $g(x) = \frac{1}{x}$

(a) Compute $(f\circ g)(x) = f(g(x)) = \tan(5g(x)) = \tan\left(\frac{5}{x}\right)$.

Then:

$$F(x) = \cos(5x)\tan\left(\frac{5}{x}\right) + 5^9$$

(b) We use the sum, product, and Chain rule.

$$F'(x) = \left(\cos(5x)\tan\left(\frac{5}{x}\right) + 5^9\right)' = \left(\cos(5x)\tan\left(\frac{5}{x}\right)\right)' + \left(5^9\right)'$$

$$= (\cos 5x)'\tan\left(\frac{5}{x}\right) + \cos(5x)\left(\tan\left(\frac{5}{x}\right)\right)' + 0,$$

and using the Chain rule:

$$= -\sin(5x)\frac{d}{dx}(5x)\tan\left(\frac{5}{x}\right) + \cos(5x)\left(\sec^2\left(\frac{5}{x}\right)\right)\cdot\left(\frac{5}{x}\right)'$$

by Chain rule.

$$= -\sin(5x)\cdot 5\cdot\tan\left(\frac{5}{x}\right) + \cos(5x)\sec^2\left(\frac{5}{x}\right)\left(-\frac{5}{x^2}\right)$$

$$\boxed{F'(x) = -5\sin(5x)\tan\left(\frac{5}{x}\right) - \frac{\cos(5x)}{x^2}\frac{1}{\cos^2\left(\frac{5}{x}\right)}}$$

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4. We use here implicit differentiation, since it is very complicated to solve the eq'n for y:

$$\frac{x^2}{4} + xy + 4y^2 = 3$$

Let us check that $(2, \frac{1}{2})$ solves the equation

$$\frac{(2)^2}{4} + 2\cdot\frac{1}{2} + 4\left(\frac{1}{2}\right)^2 = \frac{4}{4} + 2\cdot\frac{1}{2} + 4\cdot\frac{1}{4}$$
$$= 1 + 1 + 1 = 3.$$

Then, the point is on the curve.

Now by implicit differentiation:

$$\left(\frac{x^2}{4}\right)' + (xy)' + (4y^2)' = (3)'$$

$$\frac{2x}{4} + (x)'y + xy' + \underbrace{4\cdot 2y\cdot y'}_{\text{Chain rule}} = 0$$

Solving for y':

$$\frac{1}{2}x + 1\cdot y + xy' + 8yy' = 0$$

$$\frac{1}{2}x + y + (x+8y)y' = 0$$

$$\Rightarrow \boxed{y' = -\frac{\left(\frac{1}{2}x+y\right)}{x+8y}}$$ This is the implicit derivative.

At the point $(2,\frac{1}{2})$, $y'\Big|_{(2,\frac{1}{2})} = m$ is the slope of the tangent line.

$$m = y'\Big|_{(2,\frac{1}{2})} = -\frac{\left(1+\frac{1}{2}\right)}{2+8\left(\frac{1}{2}\right)} = -\frac{(2+1)/2}{(4+8)/2} = \frac{-3}{12} = -\frac{1}{4}$$

The perpendicular line has slope $m_\perp = -\frac{1}{m} = 4$

Then, the eq'n of the perp. line is

$$y - y_0 = m_\perp (x - x_0)$$

$$\boxed{y - \frac{1}{2} = 4\,(x-2)}$$

Similarly, $y = 4x - 8 + \frac{1}{2} = 4x - \frac{16}{2} + \frac{1}{2} = 4x + \left(\frac{-15}{2}\right)$

$$\Rightarrow \boxed{y = 4x - \frac{15}{2}}$$

=5=

(5) We solved this problem in class.

The position of the particle, with initial position y_0, and initial velocity. v_0 is given by:

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$$

Here, you just dropped the ball: then $v_0 = 0$ m/sec.

You are at 160 m over floor level: $y_0 = 160$ m.

Then

$$y(t) = -\frac{1}{2} g t^2 + 160.$$

The particle hits the floor when $y(t) = 0$:

$$-\frac{1}{2} g t^2 + 160 = 0.$$

$$\Rightarrow g t^2 = 2 \cdot 160 \Rightarrow$$

$$t^2 = \frac{2 \cdot 160}{g}$$

$$\boxed{t = \sqrt{\frac{2 \cdot 160}{g}} \text{ sec.}}$$

If $g = 10 \text{ m/sec}^2$: $t = \sqrt{\frac{2 \cdot 160}{10}} = \sqrt{2 \cdot 16}$

$= \sqrt{2}\sqrt{16} = 4\sqrt{2}$ sec. $\boxed{t = 4\sqrt{2} \text{ sec}}$

$\boxed{t \approx 5.64 \text{ sec}}$

The velocity is the derivative of $y(t)$

$$v(t) = y'(t) = \left(-\frac{1}{2}gt^2\right)' + (160)'$$

$$= -\left(\frac{1}{2}g\right)(2t) + 0. = -gt$$

At $t = 5.64$ sec.

v ($y'(t)$ $v(5.64) \approx -g(5.64)$

If $g \approx 10$ m/sec^2:

$$v(5.64) \approx -10(5.64) \text{ m/sec}$$

$$\boxed{v(5.64) \approx -56.4 \text{ m/sec.}}$$

The ball hits the floor at a ~~spp~~ speed of 56.4 m/sec. Very fast!

The negative sign, "−", indicates the motion is downwards

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(6) This problem is the same as problem (1), since the tangent line eq'n defines the Linearization at the point $x = 5$.

Tangent line: $y = \frac{1}{4}x - \frac{3}{4}$ Linearization at $x=5$ $\boxed{L(x) = \frac{1}{4}x - \frac{3}{4}}$

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EXAMEN # 1. – B
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PROBLEMAS

(1) **(10 puntos.)** Usando la definición de derivada, encuentre la ecuación de la recta tangente a la gráfica de

$$g(x) = \frac{1}{1+x}, \quad \text{en el punto correspondiente a} \quad x = 1.$$

(2) **(20 puntos.)** Considere las funciones $f(x) = \tan x$, $g(x) = x$, y $h(x) = -x^3 + 1$. Calcule la derivada de:

$$G(x) = \frac{f(x)}{g(x)} + \frac{h(x)}{f(x)}.$$

(3) **(20 puntos.)** Dadas $f(x) = \sqrt{x}$ y $g(x) = 7 + \dfrac{\sin x}{x}$

(a) escriba

$$G(x) = \sec(x)(f \circ g)(x) + 2^7.$$

(b) calcule la derivada de G.

(4) **(20 puntos.)** Encuentre la ecuación de la recta normal (*i.e.*, perpendicular) a la hipérbola

$$x^2/4 + 2xy - 4y^2 + x/2 = 2,$$

en el punto (2,1).

(5) **(20 puntos.)** Usted se encuentra en el edificio del Estado del Imperio a 380 metros sobre el nivel del suelo. Desde all arriba, usted suelta una pelota de esponja. ¿A qué velocidad caerá al piso? Suponga que la aceleración de gravedad es de 10 metros/segundo^2.

(6) **(10 puntos.)** Calcule la linealización de la función $g(x) = (1+x)^{-1}$, en $x = 1$. Use la definición de derivada.

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CÁLCULO DIFERENCIAL ANSWER KEY

(2) The definition of the derivative function is.

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Here, our function is: $g(x) = \frac{1}{x+1}$

Then:

$$\left(\frac{1}{x}\right)' = \lim_{\Delta x \to 0} \frac{\frac{1}{(x+1)+\Delta x} - \frac{1}{x+1}}{\Delta x} = \lim$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{(x+1) - ((x+1)+\Delta x)}{((x+1)+\Delta x)(x+1)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{((x+1) - (x+1) - \Delta x)}{+\Delta x)(x+1)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{-\Delta x}{(x+1)+\Delta x)(x+1)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{-1}{((x+1)+\Delta x)(x+1)}$$

$$= \lim_{\Delta x \to 0} -\frac{1}{(x+1)(x+1)} = -\frac{1}{(x+1)^2}$$

The slope of the tangent line to the graph $y = g(x)$

$$m = g'(1) = -\frac{1}{(1+1)^2} = -\frac{1}{2^2} = -\frac{1}{4}.$$

The eq'n of the tangent line is.

$$y - y_0 = m(x - x_0)$$

Here: $y_0 = g(x_0) = g(1) = \frac{1}{1+1} = \frac{1}{2}$.

Then $\boxed{y - \frac{1}{2} = \frac{-1}{4}(x-1)}$ point-slope form.

$$y - \frac{1}{2} = -\frac{1}{4}x + \frac{1}{4}$$

$$y = -\frac{1}{4}x + \frac{1}{4} + \frac{1}{2} = -\frac{1}{4}x + \frac{3}{4}$$

$\boxed{y = -\frac{1}{4}x + \frac{3}{4}}$ y-intercept form.

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(2) Here we use the sum, quotient and power rules

$$G'(x) = \frac{d}{dx}\left(\frac{\tan x}{x} + \frac{-x^3+1}{\tan x}\right)$$

$$= \frac{d}{dx}\left(\frac{\tan x}{x}\right) + \frac{d}{dx}\left(\frac{1-x^3}{\tan x}\right)$$

$$= \frac{x(\tan x)' - (\tan x)(x)'}{x^2} + \frac{(\tan x)(1-x^3)' - (1-x^3)(\tan x)'}{\tan^2 x}$$

$$= \frac{x\sec^2 x - (\tan x)\cdot 1}{x^2} + \frac{(\tan x)(-3x^2) - (1-x^3)\sec^2 x}{\tan^2 x}.$$

$$= \frac{x\sec^2 x - \tan x}{x^2} - \frac{3x^2\tan x + (1-x^3)\sec^2 x}{\tan^2 x}$$

$=?=$

③ let us write: $f(x)=\sqrt{x}$, $g(x) = 7 + \frac{\sin x}{x}$

$\Rightarrow$ $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)}$

$$= \sqrt{7 + \frac{\sin x}{x}}$$

Then:

$$G(x) = \sec x \sqrt{7 + \frac{\sin x}{x}} + 2^7$$

(b) Using product rule, chain rule and quotient rule.

$$G'(x) = (\sec x)' \sqrt{7 + \frac{\sin x}{x}} + \sec x \left(\sqrt{7 + \frac{\sin x}{x}}\right)' + (2^7)'$$

$$= (\sec x)(\tan x) \sqrt{7 + \frac{\sin x}{x}} + \sec x \frac{1}{2\sqrt{7 + \frac{\sin x}{x}}} \left(7 + \frac{\sin x}{x}\right)' + 0$$

$$= (\sec x)(\tan x) \sqrt{7 + \frac{\sin x}{x}} + \frac{\sec x}{2\sqrt{7 + \frac{\sin x}{x}}} \left(0 + \frac{x \cos x - \sin x}{x^2}\right)$$

$$\boxed{G'(x) = \sec x \tan x \sqrt{7 + \frac{\sin x}{x}} + \frac{\sec x}{2\sqrt{7 + \frac{\sin x}{x}}} \left(\frac{x \cos x - \sin x}{x^2}\right)}$$

④ Compute the implicit differentiation:

$$\left(\frac{x^2}{4}\right)' + 2(xy)' - 4(y^2)' + \left(\frac{x}{2}\right)' = (2)'$$

$$\frac{2x}{4} + 2(x'y + xy') - 4\underbrace{(2yy')}_{\text{Chain rule}} + \frac{1}{2} = 0$$

$$\frac{x}{2} + 2(y + xy') - 8yy' + \frac{1}{2} = 0.$$

Factoring y':

$$\frac{x}{2} + 2y + (2x - 8y)y' + \frac{1}{2} = 0$$

Solving for y': $(2x - 8y)y' = -\left(\frac{x}{2} + 2y\right)$

$$= -\left(\frac{x + 4y}{2}\right)$$

$$\Rightarrow \boxed{y' = \frac{-(x + 4y)}{2(x - 4y)}}$$ Implicit derivative.

Now, the point $(2,1)$ is on the ellipse.

$$\frac{2^2}{4} + 2(2 \cdot 1) - 4 \cdot (1)^2 + \frac{2}{2} =$$

$$= 1 + 4 - 4 + 1 =$$

$$= 2$$ ✓ Then, $(2,1)$ belongs to the ellipse

Evaluating the implicit derivative in the point (2,1).

$$m = y' = -\frac{(2+4\cdot 1)}{2(2-4\cdot 1)} = \frac{-6}{-4} = \frac{3}{2}$$

we get the slope of the tangent line.

The perpendicular line has slope:

$$m_\perp = -\frac{1}{m} = -\frac{2}{3}$$

Then, the eq'n of the perpendicular line is.

$$y - y_0 = m_\perp (x - x_0)$$

$$\boxed{y - 1 = -\frac{2}{3}(x-2)}$$ point-slope form.

Then; $y = -\frac{2}{3}x + \frac{4}{3} + 1$

$$= -\frac{2}{3}x + \frac{4+3}{3}$$

ie. $\boxed{y = -\frac{2}{3}x + \frac{7}{3}}$ is the eq'n of the perpendicular line in y-intercept form

5) The free fall position with initial velocity, v_0, and initial point y_0 is:

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$$

Since we just dropped off the ball, $v_0 = 0 \frac{m}{sec}$

The altitude is $y_0 = 380$ m. Then

$$y(t) = -\frac{1}{2} g t^2 + 380 \text{ meters.}$$

The ball hits the ~~ground~~ ground. when

$$y(t) = 0.$$

i.e. $$-\frac{1}{2} g t^2 + 380 = 0$$

$$g t^2 = 760.$$

$$t^2 = \sqrt{\frac{760}{g}} \text{ sec.}$$

Assuming, $g = 10$ m/sec^2, the ball hits the ground.

when

$$t \approx \sqrt{76} \text{ sec.}$$

then $$t \approx 8.72 \text{ sec.}$$

Now, the velocity is the derivative of the position:

$$v(t) = \dot{y}(t) = \left(-\frac{1}{2} g t^2 + 380\right)' = -\frac{1}{2} g (2t) + 0$$

$$v(t) = -gt.$$

Evaluating at $t = 8.72$ sec, and $g \approx 10\ m/sec^2$

$$v(8.72) = -(10)(8.72).$$

$$\boxed{v(8.72) = -87.2\ \frac{m}{sec}}$$ is the velocity

at which the ball would hit the floor

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(6) This problem was solved in problem (1).

We get the eq'n of the tangent line

$$y = -\frac{1}{4}x + \frac{3}{4}.$$

Then, the linearization of $g(x) = \frac{1}{1+x}$ at $x=1$ is the

function defined by the tangent line at $x=1$,

i.e.

$$\boxed{f(x) = -\frac{1}{4}x + \frac{3}{4}}$$